The Automorphisms of Affine Fusion Rings

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1. Introduction

Verlinde's formula [33]

$$V^{(g)}_{a^1\cdots a^t}=\sum_{b\in\Phi}(S_{0b})^{2(1-g)}\frac{S_{a^1b}}{S_{0b}}\cdots\frac{S_{a^tb}}{S_{0b}} \tag{1.1a}$$

arose first in rational conformal field theory (RCFT) as an extremely useful expression for the dimensions of conformal blocks on a genus g surface with t punctures. Φ here is the finite set of 'primary fields'. The matrix S comes from a representation of $\mathrm{SL}_2(\mathbb{Z})$ defined by the chiral characters of the theory. Contrary to appearances, these numbers $V^{(g)}_{\star\cdots\star}$ will always be nonnegative integers. See the excellent bibliography in [6] for references to the physics literature.

These numbers are remarkable for also arising in several other contexts: for example, as dimensions of spaces of generalised theta functions; as certain tensor product coefficients in quantum groups and Hecke algebras at roots of 1 and Chevalley groups for $\mathbb{F}_p$; as certain knot invariants for 3-manifolds; as composition laws of the superselection sectors in algebraic quantum field theories; as dimensions of spaces of intertwiners in vertex operator algebras (VOAs); in von Neumann algebras as "Connes' fusion"; in quantum cohomology; and in Lusztig's exotic Fourier transform. See for example [7,20,19,32,11,10,36,37,26], and references therein.

The more fundamental of these numbers are those corresponding to a sphere with three punctures. It is more convenient to write these in the form (called *fusion coefficients*)

$$N_{ab}^c\stackrel{\mathrm{def}}{=}V^{(0)}_{a,b,Cc}=\sum_{d\in\Phi}\frac{S_{ad}S_{bd}S^*_{cd}}{S_{0d}} \tag{1.1b}$$

where C is a permutation of Φ called *charge-conjugation* and will be defined below. The fusion coefficients uniquely determine all other Verlinde dimensions (1.1a). The symmetries of the numbers (1.1b), i.e. the permutations π of Φ obeying

$$N_{\pi a,\pi b}^{\pi c}=N_{ab}^c\ , \tag{1.2}$$

are precisely the symmetries of all numbers of the form (1.1a).

The point of introducing the N_{ab}^c in (1.1b) is that they define an algebraic structure, the *fusion ring*. Consider all formal linear combinations of objects χ_a labelled by the $a \in \Phi$; the multiplication is defined to have structure constants N_{ab}^c:

$$\chi_a \chi_b = \sum_{c \in \Phi} N_{ab}^c \chi_c \tag{1.3}$$

As an abstract ring, it is not so interesting (the fusion ring over $\mathbb{C}$ is isomorphic to $\mathbb{C}^{\|\Phi\|}$ with operations defined component-wise; over $\mathbb{Q}$ it will be a direct sum of number fields). This is analogous to the character ring of a Lie algebra, which is isomorphic as a ring to a polynomial ring. Of course it is important in both contexts that we have a preferred basis, namely $\{\chi_a\}$, and so proper definitions of isomorphisms etc. must respect that.

The most important examples of fusion rings are associated to the affine algebras, and it is to these that this paper is devoted. Their automorphisms appear explicitly for instance in the classification of modular invariant (i.e. *torus*) partition functions [17,18], and also in D-branes and boundary conditions for conformal field theory (see e.g. [1]). For instance, fusion-automorphisms (more generally, -homomorphisms) generate large classes of nonnegative integer representations of the fusion-ring, each of which is associated to a boundary (*cylinder*) partition function. This will be studied elsewhere. Also, whenever the coefficient matrix of the torus partition function is a permutation matrix (in which case the partition function is called an *automorphism invariant*), we get a fusion ring automorphism. However most torus partition functions are not automorphism invariants (although Moore-Seiberg assert that there is a sense in which any torus partition function can be interpreted as one — see e.g. [3]), and most fusion ring automorphisms do not correspond to partition functions. Nevertheless, the two problems are related. The automorphism invariants for the affine algebras were classified in [17,18]; a Lemma proved there (our Proposition 4.1 below) involving q-dimensions will be very useful to us, and conversely the arguments in Section 4 of this paper could be used to considerably simplify the proofs of [17,18].

It is surprising that it is even possible to find all affine fusion automorphisms, and in fact the arguments turn out to be rather short. It is remarkable that the answer is so simple: with few exceptions, they correspond to the Dynkin diagram symmetries.

A related task is determining which affine fusion rings are isomorphic. We answer this in section 5 below; as expected most fusion rings with different names are nonisomorphic.

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2. Generalities

2.1. The affine fusion ring

The source of some of the most interesting fusion data are the affine nontwisted Kac-Moody algebras $X_r^{(1)}$ [23]. Choose any positive integer k. Consider the (finite) set $P_+ = P_+^k(X_r^{(1)})$ of level k integrable highest weights:

$$P_+ \stackrel{\text{def}}{=} \{\sum_{j=0}^{r} \lambda_j \Lambda_j \,|\, \lambda_j \in \mathbb{Z},\ \lambda_j \geq 0,\ \sum_{j=0}^{r} a_j^\vee \lambda_j = k\} \ ,$$

where Λ_i denote the fundamental weights, and $a_j^\vee$ are the co-labels, of $X_r^{(1)}$ (the $a_j^\vee$ will be given for each algebra in §3). We will usually drop the (redundant) component $\lambda_0\Lambda_0$. Kac-Peterson [24] found a natural representation of the modular group $\mathrm{SL}_2(\mathbb{Z})$ on the complex space spanned by the affine characters χ_μ, $\mu \in P_+$: most significantly, $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is sent to the *Kac-Peterson* matrix S with entries

$$S_{\mu\nu} = c \sum_{w \in \overline{W}} \det(w)\ \exp[-2\pi \mathrm{i}\, \frac{(w(\mu+\rho)|\nu+\rho)}{\kappa}] \ . \tag{2.1a}$$

An explicit expression for the normalisation constant c is given in e.g. [23, Theorem 13.8(a)]. The inner product in (2.1a) is scaled so that the long roots have norm 2. $\overline{W}$ is the (finite) Weyl group of X_r, and acts on P_+ by fixing Λ_0. The Weyl vector ρ equals $\sum_i \Lambda_i$, and $\kappa \stackrel{\text{def}}{=} k + \sum_i a_i^\vee$. This is the matrix S appearing in (1.1); Φ there is P_+ here.

The matrix S is symmetric and unitary. One of the weights, $k\Lambda_0$, is distinguished and will be denoted '0'. It is the weight appearing in the denominator of (1.1). A useful fact is that

$$S_{\lambda 0} > 0 \qquad \text{for all } \lambda \in P_+ \ .$$

Equation (2.1a) gives us the important

$$\chi_\lambda[\mu] \stackrel{\text{def}}{=} \frac{S_{\lambda\mu}}{S_{0\mu}} = \mathrm{ch}_{\overline{\lambda}}(-2\pi \mathrm{i} \frac{\overline{\mu}+\overline{\rho}}{\kappa}) \ , \tag{2.1b}$$

where $\mathrm{ch}_{\overline{\lambda}}$ is the Weyl character of the X_r-module $L(\overline{\lambda})$. Together with the Weyl denominator formula, it provides a useful expression for the *q-dimensions*:

$$\mathcal{D}(\lambda) \stackrel{\text{def}}{=} \frac{S_{\lambda 0}}{S_{00}} = \prod_{\alpha>0} \frac{\sin(\pi\,(\lambda+\rho\,|\,\alpha)/\kappa)}{\sin(\pi\,(\rho\,|\,\alpha)/\kappa)} \ , \tag{2.1c}$$

where the product is over the positive roots $\alpha \in \overline{\Delta}_+$ of X_r. Another consequence of (2.1b) is the Kac-Walton formula (2.4).

Charge-conjugation is the order 2 permutation of P_+ given by $C\lambda = {}^t\lambda$, the weight contragredient to λ. For instance $C0 = 0$. It has the basic property that

$$S_{C\lambda,\mu} = S_{\lambda,C\mu} = S^*_{\lambda\mu} \tag{2.2a}$$

and $S^2 = C$. C corresponds to a symmetry of the (unextended) Dynkin diagram of X_r, as we will see next section.

Related to C are all the other symmetries of the unextended Dynkin diagram. We call these *conjugations*. The only $X_r^{(1)}$ with nontrivial conjugations other than charge-conjugation are $D^{(1)}_{even}$.

Another important symmetry of the matrix S is called *simple-currents*. Any weight $j \in P_+$ with q-dimension $\mathcal{D}(j) = 1$, is called a simple-current. To any such weight j is associated a permutation J of P_+ and a function $Q_j : P_+ \to \mathbb{Q}$, such that $J0 = j$ and

$$S_{J\lambda,\mu} = \exp[2\pi\mathrm{i}\, Q_j(\mu)]\, S_{\lambda\mu} \tag{2.2b}$$

The simple-currents form an abelian group, given by composition of the permutations J.

All simple-currents for the affine algebras were classified in [12], and with one unimportant exception ($E_8^{(1)}$ at level 2) correspond to symmetries of the extended Coxeter–Dynkin diagram of $X_r^{(1)}$. The simplest proof would use the methods of Proposition 4.1 below. For a more intrinsically algebraic interpretation of these simple-currents, see [25] where their group is denoted W_0^+.

Evaluating $S_{J\lambda,j'}$ in two ways gives the useful

$$Q_{j'}(J\lambda) \equiv Q_j(j') + Q_{j'}(\lambda) \qquad (\mathrm{mod}\ 1) \tag{2.2c}$$

and hence the reciprocity $Q_j(j') = Q_{j'}(j)$.

For each X_r, the inner products $(\lambda|\mu)$ of weights are rational; let N denote the least common denominator. E.g. for A_r this is $N = r+1$, while for E_8 it is $N = 1$. Choose any integer ℓ coprime to κN. Then for any $\lambda \in P_+$ there is a unique weight $\lambda^{(\ell)} \in P_+$, coroot α, and (finite) Weyl element ω such that

$$\ell\,(\lambda+\rho) = \omega(\lambda^{(\ell)} + \rho) + \kappa\alpha\ .$$

This is simply the statement that the affine Weyl orbit of $\ell\,(\lambda+\rho)$ intersects the set $P_+ + \rho$ at precisely one point (namely $\lambda^{(\ell)} + \rho$). Write $\epsilon'_\ell(\lambda) = \det\omega = \pm 1$. Then [16]

$$\epsilon'_\ell(\lambda)\, S_{\lambda^{(\ell)},\mu} = \epsilon'_\ell(\mu)\, S_{\lambda,\mu^{(\ell)}} \tag{2.3a}$$

This has an obvious interpretation as a Galois automorphism [4]: the field generated over $\mathbb{Q}$ by all entries $S_{\lambda\mu}$ lies in the cyclotomic field $\mathbb{Q}[\xi_{4N\kappa}]$ where ξ_n denotes the root of unity $\exp[2\pi\mathrm{i}/n]$; for any $\sigma_\ell \in \mathrm{Gal}(\mathbb{Q}[\xi_{4N\kappa}]/\mathbb{Q}) \cong \mathbb{Z}^\times_{4N\kappa}$, there will be a function $\epsilon_\ell : P_+ \to \{\pm 1\}$ such that

$$\sigma_\ell(S_{\lambda\mu}) = \epsilon_\ell(\lambda)\, S_{\lambda^{(\ell)},\mu} = \epsilon_\ell(\mu)\, S_{\lambda,\mu^{(\ell)}}\ . \tag{2.3b}$$

$\epsilon_\ell(\lambda)/\epsilon'_\ell(\lambda)=\sigma_\ell(c)/c$ is an unimportant sign independent of λ. This Galois action will play a fairly important role in this paper. Note that $\sigma_{-1}=C$, so this action can be thought of as a generalisation of charge-conjugation. Note also that $\sigma_\ell\circ J=J^\ell\circ\sigma_\ell$.

The fusion coefficients (1.1b) are usually computed by the Kac-Walton formula [23 p. 288, 35] (there are other codiscoverers) in terms of the tensor product multiplicities $T_{\lambda\mu}^\nu \stackrel{\text{def}}{=} \text{mult}_{L(\overline{\lambda})\otimes L(\overline{\mu})}(L(\overline{\nu}))$ in X_r:

$$N_{\lambda\mu}^\nu=\sum_{w\in W}\det(w)\,T_{\lambda\mu}^{w.\nu}\ , \tag{2.4}$$

where $w.\gamma\stackrel{\text{def}}{=}w(\gamma+\rho)-\rho$ and W is the affine Weyl group of $X_r^{(1)}$ (the dependence of $N_{\lambda\mu}^\nu$ on k arises through the action of W). We shall see shortly that these fusion coefficients, now manifestly integral, are in fact nonnegative. Let $\mathcal{R}(X_{r,k})$ denote the corresponding fusion ring.

A handy consequence of (2.4) that whenever k is large enough that $\lambda+\mu\in P_+^k(X_r^{(1)})$ (i.e. that $\sum_{i=1}^r a_i^\vee(\lambda_i+\mu_i)\le k$), then $N_{\lambda\mu}^\nu=T_{\lambda\mu}^\nu$.

It will sometimes be convenient to collect these coefficients in matrix form as the *fusion matrices* N_λ, defined by $(N_\lambda)_{\mu\nu}=N_{\lambda\mu}^\nu$. For instance, $N_0=I$ and, more generally, N_j is the permutation matrix associated to J.

The importance of (charge-)conjugation and simple-currents for us is that they respect fusions:

$$N_{C\lambda,C\mu}^{C\nu}=N_{\lambda\mu}^\nu \tag{2.5a}$$

$$N_{J\lambda,J'\mu}^{JJ'\nu}=N_{\lambda\mu}^\nu \tag{2.5b}$$

$$N_{\lambda\mu}^\nu\ne 0\ \Rightarrow\ Q_j(\lambda)+Q_j(\mu)\equiv Q_j(\nu)\qquad (\text{mod }1) \tag{2.5c}$$

for any simple-currents J,J',j.

For example, for $\mathcal{R}(A_{1,k})$ we may take $P_+=\{0,1,\ldots,k\}$ (the value of λ_1), and then the Kac-Peterson matrix is $S_{ab}=\sqrt{\frac{2}{k+2}}\,\sin(\pi\,\frac{(a+1)\,(b+1)}{k+2})$. Charge-conjugation C is trivial here, but $j=k$ is a simple-current corresponding to permutation $Ja=k-a$ and function $Q_j(a)=a/2$. The Galois action sends a to the unique weight $a^{(\ell)}\in P_+$ satisfying $a^{(\ell)}+1\equiv\pm\ell\,(a+1)$ (mod $2k+4$), where that sign there equals $\mathrm{i}^{\ell-1}\epsilon'_\ell(a)$. The fusion coefficients are given by

$$N_{ab}^c=\begin{cases}1 & \text{if } c\equiv a+b \text{ (mod 2) and } |a-b|\le c\le\min\{a+b,2k-a-b\}\\ 0 & \text{otherwise}\end{cases}\ .$$

Equation (2.4) tells us the affine fusion rules are the structure constants for the ring $\mathrm{Ch}(X_r)/\mathcal{J}_k$ where $\mathrm{Ch}(X_r)$ is the character ring for all finite-dimensional X_r-modules, and $\mathcal{J}_k$ is the subspace spanned by the elements $\mathrm{ch}_{\overline{\mu}}-(\det w)\mathrm{ch}_{\overline{w.\mu}}$. Finkelberg [8] proved that this ring is isomorphic to the K-ring of a "sub-quotient" $\widetilde{\mathcal{O}}_k$ of Kazhdan-Lusztig's category of level k integrable highest weight $X_r^{(1)}$-modules, and to Gelfand-Kazhdan's category $\widetilde{\mathcal{O}}_q$ coming from finite-dimensional modules of the quantum group U_qX_r specialised to the

root of unity $q = \xi_{2m\kappa}$ for appropriate choice of $m \in \{1,2,3\}$. They also arise from the Huang-Lepowsky coproduct [21] for the modules of the VOA $L(k,0)$. Because of these isomorphisms, we get that the $N_{\lambda\mu}^{\nu}$ do indeed lie in $\mathbb{Z}_{\geq}$, for any affine algebra.

A useful way of identifying weights in affine Weyl orbits involves computing q-dimensions and norms. Q-dimensions vary by at most a sign while norms are constant mod 2κ: $\mathcal{D}(w.\lambda) = \det(w)\,\mathcal{D}(\lambda)$ and $(w\lambda|w\lambda) \equiv (\lambda|\lambda) \pmod{2\kappa}$. The point is that for exceptional algebras at small levels, the highest weights can often be distinguished by the pair $(\mathcal{D}(\lambda), (\lambda+\rho|\lambda+\rho) \pmod{2\kappa})$. For example this is true of $E_{8,5}, E_{8,6}, F_{4,4}$. This is a useful way in practise to use both (2.4) and the Galois action (2.3).

An important property obeyed by the matrix S for any classical algebra X_r is *rank-level duality.* The first appearance of this curious duality seems to be by Frenkel [9], but by now many aspects and generalisations have been explored in the literature. For $A_r^{(1)}$, it is related to the existence of mutually commutative affine subalgbras $\widehat{\mathrm{sl}(n)}$ and $\widehat{\mathrm{sl}(k)}$ in $\widehat{\mathrm{gl}(nk)}$. Witten has another interpretation of it [37]: he found a natural map (a ring homomorphism) from the quantum cohomology of the Grassmannian $G(k,N)$, to the fusion ring of the algebra $\mathrm{u}(k) \cong \mathrm{su}(k) \oplus \mathrm{u}(1)$ at level $(N-k,N)$. Witten used the duality between $G(k,N)$ and $G(N-k,N)$ to show that the fusion rings of $\mathrm{u}(k)$ level $(N-k,N)$ and $\mathrm{u}(N-k)$ level (k,N) should coincide. A considerable generalisation, applying to any VOA (or RCFT), has been conjectured by Nahm [30], and relates to the natural involution $\sum_i [x_i] \leftrightarrow \sum_i [1-x_i]$ of torsion elements of the Bloch group.

The Kac-Peterson matrices of $\widehat{\mathrm{sl}(\ell)}$ level k and $\widehat{\mathrm{sl}(k)}$ level ℓ are related, as are those of $C_{r,k}$ and $C_{k,r}$, and $\widehat{\mathrm{so}(\ell)}$ level k and $\widehat{\mathrm{so}(k)}$ level ℓ. We will need only the symplectic one; the details will be given in §3.3.

2.2. Symmetries of fusion coefficients

DEFINITION 2.1. *By an isomorphism between fusion rings $\mathcal{R}(X_{r,k})$ and $\mathcal{R}(Y_{s,m})$ (with fusion coefficients N and M respectively) we mean a bijection $\pi : P_+^k(X_r^{(1)}) \rightarrow P_+^m(Y_s^{(1)})$ such that*

$$N_{\lambda,\mu}^{\nu} = M_{\pi\lambda,\pi\mu}^{\pi\nu} \qquad \forall\, \lambda,\mu,\nu \in P_+(X_{r,k}) \ . \tag{2.6}$$

When $X_{r,k} = Y_{s,m}$ we call π an automorphism or fusion-symmetry. *Call the pair of permutations π,π' an S-*symmetry *if*

$$S_{\pi\lambda,\pi'\mu} = S_{\lambda\mu} \qquad \forall \lambda,\mu \in P_+ \ .$$

The lemma below tells us that fusion- and S-symmetries form two isomorphic groups; the former we will label $\mathcal{A}(X_{r,k})$. Equation (2.5a) says that the charge-conjugation C, and more generally any conjugation, is a fusion-symmetry, while (2.2a) says (C,C) is an S-symmetry. Because $N_0 = I = M_{\tilde{0}}$, $N_{\lambda\mu}^0 = C_{\lambda\mu}$ and $M_{\tilde{\lambda},\tilde{\mu}}^{\tilde{0}} = \widetilde{C}_{\tilde{\lambda},\tilde{\mu}}$ (we use tilde's to denote quantities in $Y_s^{(1)}$ level m), any isomorphism π must obey $\pi 0 = \tilde{0}$ and $\widetilde{C} \circ \pi = \pi \circ C$. More generally, since N_λ is a permutation matrix of order n iff λ is a simple-current of order n, we see that an isomorphism sends simple-currents to simple-currents of equal order. We get

$$\pi(J\mu) = \pi(j)\,\pi(\mu) \ . \tag{2.7a}$$

For instance π must send J-fixed-points to $\pi(J)$-fixed-points.

More generally, a *fusion-homomorphism* π is defined in the obvious algebraic way. It turns out that for such a π, $\pi\lambda=\pi\mu$ iff $\mu=J\lambda$ for some simple-current J for which $\pi(J0)=\tilde{0}$. Moreover, $\pi(J0)=\tilde{0}$ is possible only if there are no J-fixed-points. When π is one-to-one (e.g. when there are no nontrivial simple-currents in $P_+^k(X_r^{(1)})$), then π obeys (2.6). Fusion-homomorphisms will be studied elsewhere.

The key to finding fusion-symmetries is the following Lemma.

LEMMA 2.2. *Let $\widetilde{S}$ be the Kac-Peterson matrix for $Y_s^{(1)}$ level m. Then a bijection $\pi : P_+^k(X_r^{(1)}) \rightarrow P_+^m(Y_s^{(1)})$ defines an isomorphism of fusion rings iff there exists some bijection $\pi' : P_+^k(X_r^{(1)}) \rightarrow P_+^m(Y_s^{(1)})$ such that $S_{\lambda\mu}=\widetilde{S}_{\pi\lambda,\pi'\mu}$ for all $\lambda,\mu\in P_+^k(X_r^{(1)})$. In particular, a permutation π is a fusion-symmetry iff (π,π') is an S-symmetry for some π'.*

Proof. The equality $N_{\lambda\mu}^{\nu}=M_{\pi\lambda,\pi\mu}^{\pi\nu}$ means that, for each μ, the column vectors $(\underline{x}_\mu)_\nu=\widetilde{S}_{\pi\nu,\pi\mu}$ are simultaneous eigenvectors for the fusion matrices N_λ, with eigenvalues $\widetilde{S}_{\pi\lambda,\pi\mu}/\widetilde{S}_{0,\pi\mu}$. It is easy to see from Verlinde's formula (1.1b) that any simultaneous eigenvector for all fusion matrices must be a scalar multiple of some column of S. Thus there must be a permutation π'' of $P_+^k(X_r^{(1)})$ and scalars $\alpha(\mu)$ such that $\widetilde{S}_{\pi\nu,\pi\mu}=\alpha(\mu)\,S_{\nu,\pi''\mu}$. Taking $\nu=0$ forces $\alpha(\mu)>0$, and then unitarity forces $\alpha(\mu)=1$. ■

Let π be any isomorphism, and let π' be as in the Lemma. Then π' is also an isomorphism, with $(\pi')'=\pi$. Equation (2.2b) implies for all $\lambda\in P_+$ and all simple-currents j, that

$$Q_j(\lambda)\equiv\widetilde{Q}_{\pi'j}(\pi\lambda)\equiv\widetilde{Q}_{\pi j}(\pi'\lambda)\qquad(\mathrm{mod}\ 1)\ . \tag{2.7b}$$

Another quick consequence of the Lemma is that for any Galois automorphism σ_ℓ and isomorphism π, we have $\tilde{\epsilon}_\ell(\pi\lambda)=\epsilon_\ell(\lambda)$ and $\pi(\lambda^{(\ell)})=(\pi\lambda)^{(\ell)}$. To see this, apply the invertibility of S to the equation

$$\epsilon_\ell(\lambda)\,S_{\lambda^{(\ell)},\mu}=\sigma_\ell S_{\lambda\mu}=\sigma_\ell\widetilde{S}_{\pi\lambda,\pi'\mu}=\tilde{\epsilon}_\ell(\pi\lambda)\,\widetilde{S}_{(\pi\lambda)^{(\ell)},\pi'\mu}=\tilde{\epsilon}_\ell(\pi\lambda)\,S_{\pi^{-1}(\pi\lambda)^{(\ell)},\mu}\ .$$

A very useful notion for studying the fusion ring is that of *fusion-generator*, i.e. a subset $\Gamma=\{\gamma_1,\ldots,\gamma_m\}$ of P_+ which generates $\mathcal{R}(X_{r,k})$ as a ring. Diagonalising, this is equivalent to requiring that there are m-variable polynomials $P_\lambda(x_1,\ldots,x_m)$ such that

$$\frac{S_{\lambda\mu}}{S_{0\mu}}=P_\lambda(\frac{S_{\gamma_1\mu}}{S_{0\mu}},\ldots,\frac{S_{\gamma_m\mu}}{S_{0\mu}})\qquad\forall\lambda,\mu\in P_+\ .$$

Let (π,π') be an S-symmetry, and suppose we know that $\pi\gamma=\gamma$ for all γ in the fusion-generator Γ. Then for any $\lambda\in P_+$,

$$\frac{S_{\lambda\mu}}{S_{0\mu}}=\frac{S_{\pi\lambda,\pi'\mu}}{S_{0,\pi'\mu}}=P_{\pi\lambda}(\frac{S_{\gamma_1,\pi'\mu}}{S_{0,\pi'\mu}},\ldots)=P_{\pi\lambda}(\frac{S_{\gamma_1\mu}}{S_{0\mu}},\ldots)=\frac{S_{\pi\lambda,\mu}}{S_{0\mu}}$$

for all $\mu\in P_+$, so $\pi\lambda=\lambda$.

One of the reasons fusion-symmetries for the affine algebras are so tractible is the existence of small fusion-generators. In particular, because we know that any Lie character

$\mathrm{ch}_{\overline{\mu}}$ for X_r can be written as a polynomial in the fundamental characters $\mathrm{ch}_{\Lambda_1},\ldots,\mathrm{ch}_{\Lambda_r}$, we know from (2.1b) that $\Gamma=\{\Lambda_1,\ldots,\Lambda_r\}$ is a fusion-generator for $X_r^{(1)}$ at any level k sufficiently large that P_+ contains all Λ_i (in other words, for any $k\ge\max_i a_i^\vee$). In fact, it is easy to show [18] that a fusion-generator valid for any $X_{r,k}$ is $\{\Lambda_1,\ldots,\Lambda_r\}\cap P_+$. Smaller fusion-generators usually exist — for example $\{\Lambda_1\}$ is a fusion-generator for $A_{8,k}$ whenever k is even and coprime to 3.

2.3. Standard constructions of fusion-symmetries

Simple-currents are a large source of fusion-symmetries. Let j be any simple-current of order n. Choose any number $a\in\{0,1,\ldots,n-1\}$ such that

$$\gcd(naQ_j(j)+1,n)=1\ .$$

Any solution to this defines a fusion-symmetry $\lambda\mapsto J^{naQ_j(\lambda)}\lambda$, which we shall denote $\pi[a]$ or $\pi_j[a]$. Note that from (2.2b), (2.5b) and (2.5c) that any $\pi=\pi[a]$, $a\in\mathbb{Z}$, obeys the relation $N_{\pi\lambda,\pi\mu}^{\pi\nu}=\ N_{\lambda\mu}^{\nu}$ when $N_{\lambda\mu}^{\nu}\ne 0$ (it would in fact be a fusion-endomorphism — see §2.2); the 'gcd' condition forces $\pi[a]$ to be a *permutation*. Choosing $b\equiv -a\,(naQ_j(j)+1)^{-1}$ (mod n), we find that $(\pi[a],\pi[b])$ is an S-symmetry.

When the group of simple-currents is not cyclic, this construction can be generalised in a natural way, and the resulting fusion-symmetry will be parametrised by a matrix (a_{ij}). We will meet these in §3.4.

We will call these *simple-current automorphisms*. The first examples of these were found by Bernard [2], and were generalised further in [31].

For any affine algebra $X_r^{(1)}$ and any sufficiently high level, we will see in the next section that its fusion-symmetries consist entirely of simple-current automorphisms and conjugations. For this reason, any other fusion-symmetry is called *exceptional*.

There is another general construction of fusion-symmetries, generalising C, although it yields few new examples for the affine fusion rings. If the Galois automorphism σ_ℓ is such that $0^{(\ell)}$ is a simple-current j — equivalently, that $\sigma_\ell(S_{00}^2)=S_{00}^2$ — then the permutation

$$\pi\{\ell\}:\lambda\mapsto J(\lambda^{(\ell)})$$

is a fusion-symmetry. The simplest example is $\pi\{-1\}=C$. We call $\pi\{\ell\}$ a *Galois fusion-symmetry*. A special case of these was given in [13]. To see that $\pi\{\ell\}$ works, note from

$$\epsilon_\ell(\lambda)\,S_{\lambda^{(\ell)},0}=\sigma_\ell S_{\lambda 0}=\epsilon_\ell(0)\,e^{2\pi\mathrm{i}\,Q_j(\lambda)}S_{\lambda 0}$$

that $\epsilon_\ell(\lambda)\,\epsilon_\ell(0)=e^{2\pi\mathrm{i}\,Q_j(\lambda)}$. Hence

$$S_{J\lambda^{(\ell)},\mu}=e^{2\pi\mathrm{i}\,Q_j(\mu)}\epsilon_\ell(\lambda)\,\sigma_\ell(S_{\lambda\mu})=e^{2\pi\mathrm{i}\,Q_j(\mu)}\,\epsilon_\ell(\lambda)\,\epsilon_\ell(\mu)\,S_{\lambda,\mu^{(\ell)}}=S_{\lambda,J\mu^{(\ell)}}$$

and so $(\pi\{\ell\},\pi\{\ell\}^{-1})$ is an S-symmetry. Incidentally, J will always be order 1 or 2 because $2\,Q_j(\lambda)\in\mathbb{Z}$ for all $\lambda\in P_+$.

Simple-currents (2.2), the Galois action (2.3), and the corresponding fusion-symmetries have analogues in arbitrary (i.e. not necessarily affine) fusion rings.

3. Data for the Affine Algebras.

Our main task in this paper is to find and construct all fusion-symmetries for the affine algebras $X_r^{(1)}$, for simple X_r. In this section we state the results, and in the next section we prove the completeness of our lists. Recall the simple-current automorphism $\pi[a]$ and Galois automorphism $\pi\{\ell\}$ defined in §2.3, and the notation $\kappa = k + h^\vee$. It will be convenient to write '$X_{r,k}$' for '$X_r^{(1)}$ and level k'. We write $\mathcal{S}$ for the group of symmetries of the extended Dynkin diagram.

3.1. The algebra $A_r^{(1)}$, $r \geq 1$

Define $\overline{r} = r + 1$ and $n = k + \overline{r}$. The level k highest weights of $A_r^{(1)}$ constitute the set P_+ of $\overline{r}$-tuples $\lambda = (\lambda_0, \ldots, \lambda_r)$ of non-negative integers obeying $\sum_{i=0}^r \lambda_i = k$. The Dynkin diagram symmetries form the dihedral group $\mathcal{S} = \mathfrak{D}_{r+1}$; it is generated by the charge-conjugation C and simple-current J given by $C\lambda = (\lambda_0, \lambda_r, \lambda_{r-1}, \ldots, \lambda_1)$ and $J\lambda = (\lambda_r, \lambda_0, \lambda_1, \ldots, \lambda_{r-1})$, with $Q_{J^a}(\lambda) = a\, t(\lambda)/\overline{r}$ for $t(\lambda) \stackrel{\text{def}}{=} \sum_{j=1}^r j\lambda_j$. Note that $C = id.$ for $A_1^{(1)}$.

The Kac-Peterson relation (2.1b) for $A_{r,k}$ takes the form

$$\frac{S_{\lambda\mu}}{S_{0\mu}} = \exp[-2\pi\mathrm{i}\frac{t(\lambda)\,t(\mu)}{\kappa\,\overline{r}}]\ s_{(\lambda)}(\exp[-2\pi\mathrm{i}\frac{(\mu+\rho)(1)}{\kappa}, \ldots, \exp[-2\pi\mathrm{i}\frac{(\mu+\rho)(\overline{r})}{\kappa}])\ , \qquad (3.1)$$

where $s_{(\lambda)}(x_1, \ldots, x_{r+1})$ is the Schur polynomial (see e.g. [27]) corresponding to the partition $(\lambda(1), \ldots, \lambda(\overline{r}))$, and where $\nu(\ell) = \sum_{i=\ell}^r \nu_i$ for any weight ν. In other words, $S_{\lambda\mu}/S_{0\mu}$ is the Schur polynomial corresponding to λ, evaluated at roots of 1 determined by μ.

The fusion (derived from the Pieri rule and (2.4))

$$\Lambda_1 \boxtimes \Lambda_\ell = \Lambda_{\ell+1} \boxplus (\Lambda_1 + \Lambda_\ell)\ ,$$

valid for $k \geq 2$ and $1 \leq \ell < r$, will be useful.

There are no exceptional fusion-symmetries for $A_r^{(1)}$:

Theorem 3.A. *The fusion-symmetries for $A_r^{(1)}$ level k are $C^i\pi[a]$, for $i \in \{0,1\}$ and any integer $0 \leq a \leq r$ for which $1 + ka$ is coprime to $r+1$.*

To avoid redundancies in the Theorem, for $r = 1$ or $k = 1$ take $i = 0$ only. If we write $\overline{r} = r'r''$, where r' is coprime to k and $r''|k^\infty$, then the number of simple-current automorphisms will exactly equal $r''\cdot\varphi(r')$, where φ is the Euler totient. The $\pi[a]$ commute with each other, and with C.

For example, for $A_{1,k}$ when k is odd, there is no nontrivial fusion-symmetry. When k is even, there is exactly one, sending $\lambda = \lambda_1\Lambda_1$ to λ (for λ_1 even) or $J\lambda = (k - \lambda_1)\Lambda_1$ (for λ_1 odd). For $A_{2,k}$, there are either six or four fusion-symmetries, depending on whether or not 3 divides k.

3.2. The algebra $B_r^{(1)}$, $r \geq 3$

A weight λ in P_+ satisfies $k = \lambda_0 + \lambda_1 + 2\lambda_2 + \cdots + 2\lambda_{r-1} + \lambda_r$, and $\kappa = k + 2r - 1$. The charge-conjugation is trivial, but there is a simple-current: $J\lambda = (\lambda_1, \lambda_0, \lambda_2, \ldots, \lambda_r)$. It has $Q(\lambda) = \lambda_r/2$.

The only fusion products we need are

$$\begin{aligned} \Lambda_1 \boxtimes \Lambda_i &= \Lambda_{i-1} \boxplus \Lambda_{i+1} \boxplus (\Lambda_1 + \Lambda_i) \\ \Lambda_1 \boxtimes (\ell\Lambda_r) &= (\ell\Lambda_r) \boxplus (\Lambda_1 + \ell\Lambda_r) \boxplus (\Lambda_{r-1} + (\ell-2)\Lambda_r) \end{aligned}$$

for all $1 \leq i < r-1$, $k > 2$, and $0 < \ell < k$, where we drop '$\Lambda_{r-1} + (\ell-2)\Lambda_r$' if $\ell = 1$. We will also use the character formula (2.1b)

$$\chi_{\Lambda_1}[\lambda] = \frac{S_{\Lambda_1\lambda}}{S_{0\lambda}} = 2\sum_{\ell=1}^{r} \cos(2\pi \frac{\lambda^+(\ell)}{\kappa}) + 1 \ , \tag{3.2}$$

where $\lambda^+(\ell) = (\lambda + \rho)(\ell)$ and

$$\lambda(\ell) = \sum_{i=\ell}^{r-1} \lambda_i + \frac{1}{2}\lambda_r \ .$$

For $k = 2$ ($\kappa = 2r+1$) there are several Galois fusion-symmetries — one for each Galois automorphism, since $S_{00}^2 = \frac{1}{4\kappa}$ is rational. In particular, define $\gamma^i = \gamma^{\kappa - i} = \Lambda_i$ for $i = 1, 2, \ldots, r-1$, and $\gamma^r = \gamma^{r+1} = 2\Lambda_r$. Then for any m coprime to κ, $\pi\{m\}$ fixes 0 and J, sends γ^a to γ^{ma} (where the superscript is taken mod κ), and stabilises $\{\Lambda_r, J\Lambda_r\}$ ($\pi\{m\}\Lambda_r = \Lambda_r$ iff the Jacobi symbol $(\frac{\kappa}{m})$ equals $+1$).

Why is $k = 2$ so special here? One reason is that rank-level duality associates $B_{r,2}$ with $\mathrm{u}(1)_{2r+1}$, and it is easy to confirm that $\widehat{\mathrm{u}(1)}$ has a rich variety of fusion-symmetries (and modular invariants) coming from its simple-currents. Also, the $B_{r,2}$ matrix S formally looks like the character table of the dihedral group and for some r actually equals the Kac-Peterson matrix S associated to the dihedral group $\mathfrak{D}_{\sqrt{\kappa}}$ twisted by an appropriate 3-cocycle [5] — finite group modular data tends to have significantly more modular invariants and fusion-symmetries than e.g. affine modular data.

THEOREM 3.B. *The fusion-symmetries of $B_r^{(1)}$ level k for $k \neq 2$ are $\pi[1]^i$ where $i \in \{0,1\}$. For $k = 2$ a fusion-symmetry will equal $\pi[1]^i\,\pi\{m\}$ for $i \in \{0,1\}$ and $m \in \mathbb{Z}_\kappa^\times$, $1 \leq m \leq r$.*

When $k = 1$, $\pi[1]$ is trivial. We have $\mathcal{F}(B_{r,2}) \cong \mathbb{Z}_2 \times (\mathbb{Z}_{2r+1}^\times/\{\pm 1\})$.

3.3. The algebra $C_r^{(1)}$, $r \geq 2$

A weight λ of P_+ satisfies $k = \lambda_0 + \lambda_1 + \cdots + \lambda_r$ and $\kappa = k + r + 1$. Charge-conjugation C again is trivial, and there is a simple-current J defined by $J\lambda = (\lambda_r, \lambda_{r-1}, \ldots, \lambda_1, \lambda_0)$, with $Q(\lambda) = (\sum_{j=1}^r j\lambda_j)/2$.

Choose any $\lambda \in P_+$. The Young diagram for λ is defined in the usual way: for $1 \leq \ell \leq r$, the ℓth row consists of $\lambda(\ell) \stackrel{\text{def}}{=} \sum_{i=\ell}^r \lambda_i$ boxes. Let $\tau\lambda$ denote the $C_{k,r}$ weight

whose diagram is the transpose of that for λ. (For this purpose the algebra C_1 may be identified with A_1.) For example, $\tau\Lambda_a = a\tilde{\Lambda}_1$, where we use tilde's to denote the quantities of $C_{k,r}$. In fact, $\tau : P_+(C_{r,k}) \to P_+(C_{k,r})$ is a bijection. Then

$$\tilde{S}_{\tau\lambda,\tau\mu} = S_{\lambda\mu} \ .$$

This rank-level duality for $C_r^{(1)}$ is especially interesting, as it defines a fusion ring isomorphism $\mathcal{R}(C_{r,k}) \cong \mathcal{R}(C_{k,r})$ (see §5). When $k = r$, we get a nontrivial fusion-symmetry: $\pi_{\rm rld}\lambda \stackrel{\rm def}{=} \tau\lambda$.

The only fusion product we need is

$$\Lambda_1 \boxtimes \Lambda_i = \Lambda_{i-1} \boxplus \Lambda_{i+1} \boxplus (\Lambda_1 + \Lambda_i) \ ,$$

valid for $i < r$ and $k \geq 2$. The following character formula (2.1b) will also be used:

$$\chi_{\Lambda_1}[\lambda] = \frac{S_{\Lambda_1\lambda}}{S_{0\lambda}} = 2\sum_{\ell=1}^{r} \cos(\pi \frac{\lambda^+(\ell)}{\kappa}) \ , \tag{3.3}$$

where $\lambda^+(\ell) = (\lambda + \rho)(\ell)$ as before.

THEOREM 3.C. *The fusion-symmetries for $C_r^{(1)}$ level k, when $k \neq r$ and either k or r is even, are $\pi[1]^i$ for $i \in \{0, 1\}$. When $k \neq r$ but both k and r are odd, then there is no nontrivial fusion-symmetry. When $k = r$, they are $\pi[1]^i\,\pi^j_{\rm rld}$ (k even) or $\pi[1]^i$ (k odd), for $i, j \in \{0, 1\}$.*

When $r = k$ is even, $\mathcal{A}(C_{r,k}) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

3.4. The algebra $D_r^{(1)}$, $r \geq 4$

A weight λ of P_+ satisfies $k = \lambda_0+\lambda_1+2\lambda_2+\cdots+2\lambda_{r-2}+\lambda_{r-1}+\lambda_r$, and $\kappa = k+2r-2$. For any r, there are the conjugations $C_0 = id.$ and $C_1\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_{r-2}, \lambda_r, \lambda_{r-1})$. The charge-conjugation C equals C_1 for odd r, and C_0 for even r. When $r = 4$ there are four additional conjugations; these six C_i correspond to all permutations of the $D_4^{(1)}$ Dynkin labels $\lambda_1, \lambda_3, \lambda_4$.

There are three non-trivial simple-currents, J_v, J_s and J_vJ_s. Explicitly, we have $J_v\lambda = (\lambda_1, \lambda_0, \lambda_2, \ldots, \lambda_{r-2}, \lambda_r, \lambda_{r-1})$ with $Q_v(\lambda) = (\lambda_{r-1} + \lambda_r)/2$, and

$$J_s\lambda = \begin{cases} (\lambda_r, \lambda_{r-1}, \lambda_{r-2}, \ldots, \lambda_1, \lambda_0) & \text{if } r \text{ is even,} \\ (\lambda_{r-1}, \lambda_r, \lambda_{r-2}, \ldots, \lambda_1, \lambda_0) & \text{if } r \text{ is odd,} \end{cases}$$

with $Q_s(\lambda) = (2\sum_{j=1}^{r-2} j\lambda_j - (r-2)\lambda_{r-1} - r\lambda_r)/4$. From this we compute $Q_s(J_s0) = -rk/4$.

The fusion products we need are

$$\begin{aligned} \Lambda_1 \boxtimes \Lambda_i &= \Lambda_{i-1} \boxplus \Lambda_{i+1} \boxplus (\Lambda_1 + \Lambda_i) \\ \Lambda_1 \boxtimes \Lambda_r &= \Lambda_{r-1} \boxplus (\Lambda_1 + \Lambda_r) \ , \end{aligned}$$

valid for all $1 \le i < r-2$ and $k > 2$. We also will use the character formula (2.1b)

$$\chi_{\Lambda_1}[\lambda] = \frac{S_{\Lambda_1 \lambda}}{S_{0\lambda}} = 2\sum_{\ell=1}^{r} \cos(2\pi \frac{\lambda^+(\ell)}{\kappa}) \ , \tag{3.4}$$

where $\lambda^+(\ell) = (\lambda+\rho)(\ell)$ and the orthonormal components $\lambda(\ell)$ are defined by $\lambda(\ell) = \sum_{i=\ell}^{r-1} \lambda_i + \frac{\lambda_r - \lambda_{r-1}}{2}$.

The simple-current automorphisms are as follows, and depend on whether r and k are even or odd. When r is odd, the group of simple-currents is generated by J_s. If in addition k is odd, there will be only two simple-current automorphisms: $\pi = \pi' = \pi[a] = J_s^{4aQ_s}$ for $a \in \{0,2\}$. If instead k is even, there will be four simple-current automorphisms: $\pi = \pi[a]$ and $\pi' = \pi[ak-a]$ for $0 \le a \le 3$. When $k \equiv 2 \pmod 4$, these form the group $\mathbb{Z}_2 \times \mathbb{Z}_2$, otherwise when $4|k$ the group is $\mathbb{Z}_4$.

When r is even, the simple-currents are generated by both J_v and J_s. If in addition k is even, we have 16 simple-current automorphisms:

$$\pi = \pi \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \text{and} \qquad \pi' = \pi \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

for any $a,b,c,d \in \{0,1\}$, forming a group isomorphic to $\mathbb{Z}_2^4$. This notation means

$$\pi \begin{bmatrix} a & b \\ c & d \end{bmatrix} (\lambda) \stackrel{\text{def}}{=} J_v^{2a\,Q_v(\lambda)+2b\,Q_s(\lambda)} J_s^{2c\,Q_v(\lambda)+2d\,Q_s(\lambda)} \lambda \ . \tag{3.5}$$

When k is odd, we will have six simple-current automorphisms:

$$\begin{aligned} \pi \ &= \pi \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \qquad \text{with} \qquad \pi' = \pi \begin{bmatrix} a\,(d+1) & \frac{dr}{2} \\ \frac{dr}{2} & d \end{bmatrix} \\ \text{or} \quad \pi \ &= \pi \begin{bmatrix} \frac{r}{2}+1 & b \\ c & 1 \end{bmatrix} \qquad \text{with} \qquad \pi' = \pi \begin{bmatrix} \frac{r}{2}+1+bc\frac{r}{2} & b+\frac{r}{2} \\ \frac{r}{2}+1+bc+b & 1 \end{bmatrix} , \end{aligned}$$

where $a = \frac{r}{2}$ or $d = 0$, and where $b = 1$ or $d = 1$. The corresponding permutation of P_+ is still given by (3.5). Again, for these r,k, these are the values of a,b,c,d for which (3.5) is invertible. For k odd, the group of simple-current automorphisms is isomorphic to the symmetric group $\mathfrak{S}_3$ when 4 divides r, and to $\mathbb{Z}_6$ when $r \equiv 2 \pmod 4$.

For $k = 2$ (so $\kappa = 2r$), there are several Galois fusion-symmetries. In particular, write $\lambda^i = \lambda^{2r-i} = \Lambda_i$ for $1 \le i \le r-2$, and $\lambda^{r\pm 1} = \Lambda_{r-1} + \Lambda_r$. As with $B_{r,2}$, $S_{00}^2 = \frac{1}{4\kappa}$ is rational so for any m coprime to $2r$, we get a Galois fusion-symmetry $\pi\{m\}$. It takes λ^a to λ^{ma}, where the superscript is taken mod $2r$, and will fix $J_v 0$. Also, $\pi\{m\}$ will send $J_s 0$ to $J_s^m 0$, as well as stabilise the set $\{\Lambda_r, \Lambda_{r-1}, J_v\Lambda_r, J_v\Lambda_{r-1}\}$. (In particular, put $t = r$ when r is even or when $m \equiv 1 \pmod 4$, otherwise put $t = r-1$; then for any i,j, $\pi\{m\}\, C_1^j J_v^i \Lambda_r$ is $C_1^j J_v^i \Lambda_t$ or $C_1^j J_v^{i+1} \Lambda_t$, when the Jacobi symbol $(\frac{\kappa}{m})$ is ± 1, respectively.)

THEOREM 3.D. *The fusion-symmetries of* $D_r^{(1)}$ *for* $k \ne 2$ *are all of the form* $C_i\,\pi$, *where* C_i *is a conjugation, and where* π *is a simple-current automorphism. Similarly for*

$D_4^{(1)}$ *at* $k=2$. *Finally, when both* $k=2$ *and* $r>4$, *any fusion-symmetry* π *can be written as* $\pi = C_1^a\,\pi_v^b\,\pi\{m\}$ *for* $a,b\in\{0,1\}$ *and any* $m\in\mathbb{Z}_{2r}^\times$, $1\le m<r$.

π_v here refers to the simple-current automorphism $\pi[2]$ or $\pi[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}]$, for r odd/even. When $k=1$, $\mathcal{A}(D_{even,1})\cong\mathfrak{S}_3$, corresponding to any permutation of $\Lambda_1,\Lambda_{r-1},\Lambda_r$, and $\mathcal{A}(D_{odd,1})=\langle C_1\rangle\cong\mathbb{Z}_2$. When $r>4$, $\mathcal{A}(D_{r,2})\cong(\mathbb{Z}_{2r}^\times/\{\pm1\})\times\mathbb{Z}_2\times\mathbb{Z}_2$ or $\mathbb{Z}_r^\times\times\mathbb{Z}_2$ for r even/odd. $\mathcal{A}(D_{4,2})$ has 24 elements, and any element can be written uniquely as $C_i\,\pi\begin{bmatrix}a&0\\0&d\end{bmatrix}$.

3.5. The algebra $E_6^{(1)}$

A weight λ of P_+ satisfies $k=\lambda_0+\lambda_1+2\lambda_2+3\lambda_3+2\lambda_4+\lambda_5+2\lambda_6$ and $\kappa=k+12$. The charge-conjugation acts as $C\lambda=(\lambda_0,\lambda_5,\lambda_4,\lambda_3,\lambda_2,\lambda_1,\lambda_6)$. The order 3 simple-current J is given by $J\lambda=(\lambda_5,\lambda_0,\lambda_6,\lambda_3,\lambda_2,\lambda_1,\lambda_4)$ with $Q(\lambda)=(-\lambda_1+\lambda_2-\lambda_4+\lambda_5)/3$.

The fusion products we need can be derived from [29] using (2.4):

$$\Lambda_1\boxtimes\Lambda_1=(\Lambda_2)_2\boxplus(\Lambda_5)_1\boxplus(2\Lambda_1)_2 \tag{3.6a}$$

$$\Lambda_1\boxtimes\Lambda_5=(0)_1\boxplus(\Lambda_6)_2\boxplus(\Lambda_1+\Lambda_5)_2 \tag{3.6b}$$

$$\Lambda_1\boxtimes\Lambda_2=(\Lambda_3)_3\boxplus(\Lambda_6)_2\boxplus(\Lambda_1+\Lambda_2)_3\boxplus(\Lambda_1+\Lambda_5)_2 \tag{3.6c}$$

$$\Lambda_1\boxtimes(2\Lambda_1)=(3\Lambda_1)_3\boxplus(\Lambda_1+\Lambda_2)_3\boxplus(\Lambda_1+\Lambda_5)_2 \tag{3.6d}$$

where the outer subscript on any summand denotes the smallest level where that summand appears (it will also appear at all larger levels). So for example $\Lambda_1\boxtimes\Lambda_1$ equals $\Lambda_2\boxplus\Lambda_5\boxplus(2\Lambda_1)$ for any $k\ge2$, but equals Λ_5 at $k=1$. A similar convention is used in (3.7) and elsewhere for higher fusion multiplicities (the number of subscripts used will equal the numerical value of the fusion coefficient).

THEOREM 3.E6. *The fusion-symmetries of* $E_6^{(1)}$ *are* $C^i\,\pi[a]$, *for any* $i\in\{0,1\}$ *and any* $a\in\{0,1,2\}$ *for which* $ak\not\equiv1$ *(mod 3).*

3.6. The algebra $E_7^{(1)}$

A weight λ in P_+ satisfies $k=\lambda_0+2\lambda_1+3\lambda_2+4\lambda_3+3\lambda_4+2\lambda_5+\lambda_6+2\lambda_7$, and $\kappa=k+18$. The charge-conjugation is trivial, but there is a simple-current J given by $J\lambda=(\lambda_6,\lambda_5,\ldots,\lambda_1,\lambda_0,\lambda_7)$. It has $Q(\lambda)=(\lambda_4+\lambda_6+\lambda_7)/2$.

The only fusion products we need can be obtained from [29] and (2.4):

$$\begin{aligned}
\Lambda_6\boxtimes\Lambda_6&=(0)_1\boxplus(\Lambda_1)_2\boxplus(\Lambda_5)_2\boxplus(2\Lambda_6)_2\\
\Lambda_1\boxtimes\Lambda_6&=(\Lambda_6)_2\boxplus(\Lambda_7)_2\boxplus(\Lambda_1+\Lambda_6)_3\\
\Lambda_5\boxtimes\Lambda_6&=(\Lambda_4)_3\boxplus(\Lambda_6)_2\boxplus(\Lambda_7)_2\boxplus(\Lambda_1+\Lambda_6)_3\boxplus(\Lambda_5+\Lambda_6)_3\\
\Lambda_6\boxtimes(2\Lambda_6)&=(\Lambda_6)_2\boxplus(\Lambda_1+\Lambda_6)_3\boxplus(3\Lambda_6)_3\boxplus(\Lambda_5+\Lambda_6)_3\\
\Lambda_4\boxtimes\Lambda_6&=(\Lambda_2)_3\boxplus(\Lambda_3)_4\boxplus(\Lambda_5)_3\boxplus(\Lambda_1+\Lambda_5)_4\boxplus(\Lambda_4+\Lambda_6)_4\boxplus(\Lambda_6+\Lambda_7)_3\\
\Lambda_6\boxtimes\Lambda_7&=(\Lambda_1)_2\boxplus(\Lambda_2)_3\boxplus(\Lambda_5)_2\boxplus(\Lambda_6+\Lambda_7)_3\\
\Lambda_6\boxtimes(\Lambda_5+\Lambda_6)&=(\Lambda_5)_3\boxplus(2\Lambda_5)_4\boxplus(2\Lambda_6)_3\boxplus(\Lambda_6+\Lambda_7)_3\boxplus(\Lambda_1+\Lambda_5)_4\\
&\quad\boxplus(\Lambda_4+\Lambda_6)_4\boxplus(\Lambda_1+2\Lambda_6)_4\boxplus(\Lambda_5+2\Lambda_6)_4
\end{aligned}$$

At $k=3$ there is an order 3 Galois fusion-symmetry $\pi_3=\pi\{5\}$, which sends $J^i\Lambda_1\mapsto J^i(2\Lambda_6)\mapsto J^i\Lambda_2\mapsto J^i\Lambda_1$ and fixes the other six weights.

THEOREM 3.E7. *The only nontrivial fusion-symmetries for $E_7^{(1)}$ are $\pi[1]$ at even k, as well as π_3 and its inverse at $k=3$.*

3.7. The algebra $E_8^{(1)}$

A weight λ in P_+ satisfies $k=\lambda_0+2\lambda_1+3\lambda_2+4\lambda_3+5\lambda_4+6\lambda_5+4\lambda_6+2\lambda_7+3\lambda_8$, and $\kappa=k+30$. The conjugations and simple-currents are all trivial, except for an anomolous simple-current at $k=2$, sending $P_+=(0,\Lambda_1,\Lambda_7)$ to $(\Lambda_7,\Lambda_1,0)$, which plays no role in this paper (except in Theorem 5.1).

The only fusion products we need can be derived from [28] and (2.4):

$$\Lambda_1\boxtimes\Lambda_1=(0)_2\boxplus(\Lambda_1)_3\boxplus(\Lambda_2)_3\boxplus(\Lambda_7)_2\boxplus(2\Lambda_1)_4 \tag{3.7a}$$

$$\begin{aligned}\Lambda_2\boxtimes\Lambda_2=\,&(0)_3\boxplus(\Lambda_1)_4\boxplus 2\boxdot(\Lambda_2)_{34}\boxplus 2\boxdot(\Lambda_3)_{45}\boxplus(\Lambda_4)_5\boxplus(\Lambda_6)_4\boxplus 2\boxdot(\Lambda_7)_{34}\\&\boxplus 2\boxdot(\Lambda_8)_{44}\boxplus 3\boxdot(\Lambda_1+\Lambda_7)_{445}\boxplus 2\boxdot(2\Lambda_1)_{45}\boxplus(2\Lambda_2)_6\boxplus(2\Lambda_7)_4\boxplus 2\boxdot(\Lambda_1+\Lambda_2)_{55}\\&\boxplus(\Lambda_1+\Lambda_3)_6\boxplus 2\boxdot(\Lambda_1+\Lambda_8)_{55}\boxplus(\Lambda_2+\Lambda_7)_5\boxplus(2\Lambda_1+\Lambda_7)_6\boxplus(3\Lambda_1)_6\end{aligned} \tag{3.7b}$$

$$\begin{aligned}\Lambda_7\boxtimes\Lambda_7=\,&(0)_2\boxplus(\Lambda_1)_3\boxplus(\Lambda_2)_3\boxplus(\Lambda_3)_4\boxplus(\Lambda_6)_4\boxplus(\Lambda_7)_3\boxplus(\Lambda_8)_3\\&\boxplus(2\Lambda_1)_4\boxplus(2\Lambda_7)_4\boxplus(\Lambda_1+\Lambda_7)_4\end{aligned} \tag{3.7c}$$

$$\begin{aligned}(2\Lambda_1)\boxtimes(2\Lambda_1)=\,&(0)_4\boxplus(\Lambda_1)_5\boxplus(\Lambda_2)_5\boxplus(\Lambda_3)_4\boxplus(\Lambda_7)_4\boxplus 2\boxdot(2\Lambda_1)_{46}\boxplus(2\Lambda_2)_6\\&\boxplus(2\Lambda_7)_4\boxplus 2\boxdot(\Lambda_1+\Lambda_2)_{56}\boxplus(\Lambda_1+\Lambda_7)_5\boxplus(\Lambda_2+\Lambda_7)_5\boxplus(3\Lambda_1)_7\\&\boxplus(2\Lambda_1+\Lambda_2)_7\boxplus(2\Lambda_1+\Lambda_7)_6\boxplus(4\Lambda_1)_8\end{aligned} \tag{3.7d}$$

$$\begin{aligned}\Lambda_1\boxtimes\Lambda_4=\,&(\Lambda_3)_5\boxplus(\Lambda_4)_6\boxplus(\Lambda_5)_6\boxplus(\Lambda_6)_5\boxplus(\Lambda_1+\Lambda_3)_6\boxplus(\Lambda_1+\Lambda_4)_7\boxplus(\Lambda_1+\Lambda_6)_6\\&\boxplus(\Lambda_1+\Lambda_8)_5\boxplus(\Lambda_2+\Lambda_7)_5\boxplus(\Lambda_7+\Lambda_8)_5\boxplus(\Lambda_2+\Lambda_8)_6\boxplus(\Lambda_3+\Lambda_7)_6\end{aligned} \tag{3.7e}$$

$$\begin{aligned}\Lambda_1\boxtimes(\Lambda_1+\Lambda_3)=\,&(\Lambda_3)_6\boxplus(\Lambda_4)_6\boxplus(\Lambda_1+\Lambda_2)_6\boxplus 2\boxdot(\Lambda_1+\Lambda_3)_{67}\boxplus(\Lambda_1+\Lambda_4)_7\\&\boxplus(\Lambda_1+\Lambda_6)_6\boxplus(\Lambda_1+\Lambda_8)_6\boxplus(\Lambda_2+\Lambda_3)_7\boxplus(\Lambda_2+\Lambda_7)_6\boxplus(2\Lambda_2)_6\\&\boxplus(\Lambda_2+\Lambda_8)_6\boxplus(\Lambda_3+\Lambda_7)_6\boxplus(2\Lambda_1+\Lambda_8)_7\boxplus(2\Lambda_1+\Lambda_2)_7\\&\boxplus(2\Lambda_1+\Lambda_3)_8\boxplus(2\Lambda_1+\Lambda_7)_6\boxplus(\Lambda_1+\Lambda_2+\Lambda_7)_7\end{aligned} \tag{3.7f}$$

$$\Lambda_1\boxtimes(2\Lambda_7)=(\Lambda_6)_4\boxplus(\Lambda_1+\Lambda_7)_4\boxplus(2\Lambda_7)_5\boxplus(\Lambda_2+\Lambda_7)_5\boxplus(\Lambda_7+\Lambda_8)_5\boxplus(\Lambda_1+2\Lambda_7)_6 \tag{3.7g}$$

A fusion-symmetry at $k=4$, called π_4, was first found in [15]. It interchanges $\Lambda_1\leftrightarrow\Lambda_6$ and fixes the other eight weights in P_+. There also is a fusion-symmetry, called π_5, at $k=5$ which interchanges $\Lambda_7\leftrightarrow 2\Lambda_1$, $\Lambda_8\leftrightarrow\Lambda_1+\Lambda_2$, and $\Lambda_6\leftrightarrow\Lambda_2+\Lambda_7$, and fixes the nine other weights. The exceptional π_5 is closely related to the Galois permutation $\lambda\mapsto\lambda^{(13)}$.

THEOREM 3.E8. *The only nontrivial fusion-symmetries for $E_8^{(1)}$ are π_4 and π_5, occurring at $k=4$ and 5 respectively.*

3.8. The algebra $F_4^{(1)}$

A weight λ in P_+ satisfies $k=\lambda_0+2\lambda_1+3\lambda_2+2\lambda_3+\lambda_4$, and $\kappa=k+9$. Again, the conjugations and simple-currents are trivial.

There are Galois fusion-symmetries at levels $k = 3$ and 4. In particular, for $k = 3$ we have the fusion-symmetry $\pi_3 = \pi\{5\}$ which interchanges both $\Lambda_2 \leftrightarrow \Lambda_4$ and $\Lambda_1 \leftrightarrow 3\Lambda_4$, and fixes the other five weights in P_+. The exceptional π_3 was found independently in [34,14]. For $k = 4$ we get a fusion-symmetry of order 4, which we will call π_4. It fixes 0, $\Lambda_2 + \Lambda_4$, $\Lambda_3 + \Lambda_4$, and $2\Lambda_4$, and permutes $\Lambda_4 \mapsto \Lambda_1 \mapsto 2\Lambda_1 \mapsto 4\Lambda_4 \mapsto \Lambda_4$, $\Lambda_2 \mapsto 2\Lambda_3 \mapsto 3\Lambda_4 \mapsto \Lambda_3 \mapsto \Lambda_2$, and $\Lambda_1+\Lambda_3 \mapsto \Lambda_3+2\Lambda_4 \mapsto \Lambda_1+\Lambda_4 \mapsto \Lambda_1+2\Lambda_4 \mapsto \Lambda_1+\Lambda_3$. Its square π_4^2 equals the fusion-symmetry $\pi\{5\}$.

The only fusion products we need can be obtained from [29] and (2.4):

$$\begin{aligned}
\Lambda_4 \boxtimes \Lambda_4 &= (0)_1 \boxplus (\Lambda_1)_2 \boxplus (\Lambda_3)_2 \boxplus (\Lambda_4)_1 \boxplus (2\Lambda_4)_2 \\
\Lambda_1 \boxtimes \Lambda_4 &= (\Lambda_3)_2 \boxplus (\Lambda_4)_2 \boxplus (\Lambda_1 + \Lambda_4)_3 \\
\Lambda_3 \boxtimes \Lambda_4 &= (\Lambda_1)_2 \boxplus (\Lambda_2)_3 \boxplus (\Lambda_3)_2 \boxplus (\Lambda_4)_2 \boxplus (\Lambda_1 + \Lambda_4)_3 \boxplus (\Lambda_3 + \Lambda_4)_3 \boxplus (2\Lambda_4)_2 \\
(2\Lambda_4) \boxtimes \Lambda_4 &= (\Lambda_3)_2 \boxplus (\Lambda_4)_2 \boxplus (2\Lambda_4)_2 \boxplus (3\Lambda_4)_3 \boxplus (\Lambda_1 + \Lambda_4)_3 \boxplus (\Lambda_3 + \Lambda_4)_3
\end{aligned}$$

Theorem 3.F4. *The only nontrivial fusion-symmetries of $F_4^{(1)}$ are π_3 at level 3, and π_4^i for $1 \le i \le 3$, which occur at level 4.*

3.9. The algebra $G_2^{(1)}$

A weight λ in P_+ satisfies $k = \lambda_0 + 2\lambda_1 + \lambda_2$, and $\kappa = k + 4$. The conjugations and simple-currents are all trivial.

Again there are nontrivial Galois fusion-symmetries. At $k = 3$, we have the order 3 fusion-symmetry $\pi_3 = \pi\{4\}$ sending $\Lambda_1 \mapsto 3\Lambda_2 \mapsto \Lambda_2 \mapsto \Lambda_1$, and fixing the remaining three weights. It was found in [14]. At $k = 4$, we have $\pi_4 = \pi\{5\}$ permuting both $\Lambda_1 \leftrightarrow 4\Lambda_2$ and $2\Lambda_1 \leftrightarrow \Lambda_2$, and fixing the other five weights. It was found independently in [34,14], and in §5 we will see that it is closely related to the π_3 of $F_{4,3}$.

The only fusion products we will need can be obtained from [29] and (2.4):

$$\begin{aligned}
\Lambda_2 \boxtimes \Lambda_2 &= (0)_1 \boxplus (\Lambda_1)_2 \boxplus (\Lambda_2)_1 \boxplus (2\Lambda_2)_2 \\
\Lambda_2 \boxtimes \Lambda_2 \boxtimes \Lambda_2 &= (0)_1 \boxplus 2 \boxdot (\Lambda_1)_{22} \boxplus 4 \boxdot (\Lambda_2)_{1122} \boxplus 3 \boxdot (2\Lambda_2)_{222} \boxplus 2 \boxdot (\Lambda_1 + \Lambda_2)_{33} \boxplus (3\Lambda_2)_3
\end{aligned}$$

Theorem 3.G2. *The only nontrivial fusion-symmetries for $G_2^{(1)}$ are $(\pi_3)^{\pm 1}$ at $k = 3$, and π_4 at $k = 4$.*

4. The Arguments

The fundamental reason the classification of fusion-symmetries for the affine algebras is so accessible is (2.1b), which reduces the problem to studying Lie group characters at elements of finite order. These values have been studied by a number of people — see e.g. [22,28] — and the resulting combinatorics is often quite pretty.

Lemma 2.2 implies that a fusion-symmetry π preserves q-dimensions: $\mathcal{D}(\lambda) = \mathcal{D}(\pi\lambda)$ $\forall \lambda \in P_+$. In this subsection we use that to find a weight $\Lambda_\star$ for each algebra which must be essentially fixed by π.

4.1. q-dimensions

The most basic properties obeyed by the q-dimensions $\mathcal{D}(\lambda)=\frac{S_{\lambda 0}}{S_{00}}$ are that $\mathcal{D}(\lambda)\geq 1$, and $\mathcal{D}(s\lambda)=\mathcal{D}(\lambda)$ for any $s\in\mathcal{S}$. Recall that $\mathcal{S}$ is the symmetry group of the extended Dynkin diagram of $X_r^{(1)}$, and that $s\in\mathcal{S}$ acts on P_+ by permuting the Dynkin labels.

The argument yielding Proposition 4.1 below relies heavily on the following observation. Use (2.1c) to extend the domain of $\mathcal{D}$ from P_+ to the fundamental chamber C_+:

$$C_+\stackrel{\text{def}}{=}\{\sum_{i=0}^r x_i\Lambda_i\,|\,x_i\in\mathbb{R},\ x_i>-1,\ \sum_{i=0}^r x_i a_i^\vee=k\}\ .$$

Choose any $a,b\in C_+$. Then a straightforward calculation from (2.1c) gives

$$\frac{d}{dt}\mathcal{D}(ta+(1-t)b)=0\quad\Longrightarrow\quad\frac{d^2}{dt^2}\mathcal{D}(ta+(1-t)b)<0$$

for $0<t<1$. This means that for all $0<t<1$,

$$\mathcal{D}(ta+(1-t)b)>\min\{\mathcal{D}(a),\,\mathcal{D}(b)\}\ . \tag{4.1}$$

Proposition 4.1 [17,18]. *For the following algebras $X_r^{(1)}$ and levels k, and choices of weight $\Lambda_\star$, $\mathcal{D}(\lambda)=\mathcal{D}(\Lambda_\star)$ implies $\lambda\in\mathcal{S}\Lambda_\star$:*

(a) *For $A_r^{(1)}$ any level k, where $\Lambda_\star=\Lambda_1$;*
(b) *For $B_r^{(1)}$ any level $k\neq 2$, where $\Lambda_\star=\Lambda_1$;*
(c) *For $C_r^{(1)}$ any level k (except for $(r,k)=(2,3)$ or $(3,2)$), where $\Lambda_\star=\Lambda_1$;*
(d) *For $D_r^{(1)}$ any level $k\neq 2$, where $\Lambda_\star=\Lambda_1$;*
(e6) *For $E_6^{(1)}$ any level k, where $\Lambda_\star=\Lambda_1$;*
(e7) *For $E_7^{(1)}$ any level $k\neq 3$, where $\Lambda_\star=\Lambda_6$;*
(e8) *For $E_8^{(1)}$ any level $k\neq 1,4$, where $\Lambda_\star=\Lambda_1$;*
(f4) *For $F_4^{(1)}$ any level $k\neq 3,4$, where $\Lambda_\star=\Lambda_4$;*
(g2) *For $G_2^{(1)}$ level any $k\neq 3,4$, where $\Lambda_\star=\Lambda_2$.*

The missing cases are: $B_{r,2}$ where $\mathcal{D}(\Lambda_1)=\mathcal{D}(\Lambda_2)=\cdots=\mathcal{D}(\Lambda_{r-1})=\mathcal{D}(2\Lambda_r)$;
$D_{r,2}$ where $\mathcal{D}(\Lambda_1)=\cdots=\mathcal{D}(\Lambda_{r-2})$;
$C_{2,3}$ where $\mathcal{D}(\Lambda_2)=\mathcal{D}(3\Lambda_1)=\mathcal{D}(\Lambda_1)$, and its rank-level dual $C_{3,2}$;
$E_{7,3}$ where $\mathcal{D}(\Lambda_1)=\mathcal{D}(\Lambda_2)=\mathcal{D}(\Lambda_6)$;
$E_{8,1}$ where $\Lambda_1\notin P_+=\{0\}$, and $E_{8,4}$ where $\mathcal{D}(\Lambda_1)=\mathcal{D}(\Lambda_6)$;
$F_{4,3}$ where $\mathcal{D}(\Lambda_2)=\mathcal{D}(\Lambda_4)$, and $F_{4,4}$ where $\mathcal{D}(\Lambda_1)=\mathcal{D}(2\Lambda_1)=\mathcal{D}(4\Lambda_4)=\mathcal{D}(\Lambda_4)$;
$G_{2,3}$ where $\mathcal{D}(\Lambda_1)=\mathcal{D}(\Lambda_2)=\mathcal{D}(3\Lambda_2)$, and $G_{2,4}$ where $\mathcal{D}(\Lambda_2)=\mathcal{D}(2\Lambda_1)$.

The weight $\Lambda_\star$ singled out by Proposition 4.1 (i.e. $\Lambda_\star=\Lambda_1$ for $A_r^{(1)}$, ..., $\Lambda_\star=\Lambda_2$ for $G_2^{(1)}$) is the nonzero weight with smallest Weyl dimension. What we find is that, for all but the smallest levels (see [18, Table 3]), $\Lambda_\star$ will also have the smallest q-dimension after the simple-currents.

The complete proof of Proposition 4.1 is given in [18], but to illustrate the ideas we will sketch here the most interesting ($A_r^{(1)}$) and the most difficult ($E_8^{(1)}$) cases.

Consider first $A_{r,k}$. By choosing $a-b=\Lambda_i-\Lambda_j$ in (4.1), we get that either $\lambda=k\Lambda_\ell$ for some ℓ, in which case λ is a simple-current and (for $k\neq 1$) $\mathcal{D}(\lambda)<\mathcal{D}(\Lambda_1)$, or $\mathcal{D}(\lambda)\geq\mathcal{D}(\Lambda_\ell)$ for some ℓ, with equality iff $\lambda\in\mathcal{S}\Lambda_\ell$. But then rank-level duality $A_{r,k}\leftrightarrow A_{k-1,r+1}$ (defined as for $C_{r,k}$, and which is exact for $A_{r,k}$ q-dimensions) and (4.1) with $a-b=\widetilde{\Lambda_0}-\widetilde{\Lambda_1}$ give us $\mathcal{D}(\Lambda_\ell)=\widetilde{\mathcal{D}}(\ell\widetilde{\Lambda_1})\geq\widetilde{\mathcal{D}}(\widetilde{\Lambda_1})=\mathcal{D}(\Lambda_1)$, with equality iff $\ell=1$ or r. Combining these results yields Proposition 4.1(a).

For $E_{8,k}$, run through each $a-b=a_j^\vee\Lambda_i-a_i^\vee\Lambda_j$ to reduce the proof to comparing $\mathcal{D}(\Lambda_1)$ with $\mathcal{D}(\frac{k}{a_i^\vee}\Lambda_i)$ for $i\neq 0$, or $\mathcal{D}(\Lambda_i)$ for $i\neq 0,1$ (the argument in [18] unnecessarily complicated things by restricting to integral weights). Standard arguments (see [18] for details) quickly show that the q-dimension $\mathcal{D}(\frac{k}{a_i^\vee}\Lambda_i)$ monotonically increases with k to ∞, while $\mathcal{D}(\Lambda_i)$ monotonically increases with k to the Weyl dimension of Λ_i. The proof of Proposition 4.1(e8) then reduces to a short computation.

4.2. The A-series argument

Recall that $\overline{r}=r+1$. Proposition 4.1(a) tells us that $\pi\Lambda_1=C^aJ^b\Lambda_1$, for some a,b. Hitting π with C^a, we can assume without loss of generality that $a=0$. Write $\pi(J0)=J^c0$; then π can be a permutation of P_+ only if c is coprime to $\overline{r}$.

If $k=1$ then $P_+=\{0,J0,\dots,J^r0\}$ so $\pi=\pi[c-1]$. Thus we can assume $k\geq 2$.

Useful is the coefficient of λ in the tensor product $\Lambda_1\otimes\cdots\otimes\Lambda_1$ (ℓ times): it is 0 unless $t(\lambda)=\ell$, in which case the coefficient is $\frac{\ell!}{h(\lambda)}$ (to get this, compare (3.1) above with [27, p.114]) — we equate here the fundamental weights $\Lambda_{\overline{r}}$ and Λ_0, so e.g. '$\frac{k}{\overline{r}}\Lambda_{\overline{r}}$' equals '0' when $\overline{r}$ divides k. Here, $h(\lambda)=\prod h(x)$ is the product of the hook-lengths of the Young diagram corresponding to λ. Equation (2.4) tells us that as long as $t(\lambda)=\ell\leq k$, the number $\frac{\ell!}{h(\lambda)}$ will also be the coefficient of N_λ in the fusion power $(N_{\Lambda_1})^\ell$. Note that $J0=k\Lambda_1$ is the only simple-current appearing in the fusion product $\Lambda_1\boxtimes\cdots\boxtimes\Lambda_1$ (k times). Thus the only nontrivial simple-current appearing in the fusion $\pi\Lambda_1\boxtimes\cdots\boxtimes\pi\Lambda_1$ will be $J^{bk}J0$ (0 will appear iff $\overline{r}$ divides k). Hence $bk+1\equiv c \pmod{\overline{r}}$ must be coprime to $\overline{r}$. This is precisely the condition needed for $\pi[b]$ to be a simple-current automorphism.

In other words, it suffices to consider $\pi\Lambda_1=\Lambda_1$ and hence $\pi[J0]=J0$. We are done if $r=1$, so assume $r\geq 2$. From the $\Lambda_1\boxtimes\Lambda_1$ fusion, we get that $\pi\Lambda_2\in\{\Lambda_2,2\Lambda_1\}$. Note that $k\Lambda_1$ occurs (with multiplicity 1) in the tensor and fusion product of $2\Lambda_1$ with $k-2$ Λ_1's, but that it doesn't in the tensor (hence fusion) product of Λ_2 with $k-2$ Λ_1's (recall that $k\Lambda_1\succ(k-2)\Lambda_1+\Lambda_2$ in the usual partial order on weights). Since $\Lambda_2\boxtimes\Lambda_1\boxtimes\cdots\boxtimes\Lambda_1$ does not contain $J0$, $(\pi\Lambda_2)\boxtimes(\pi\Lambda_1)\boxtimes\cdots\boxtimes(\pi\Lambda_1)$ should also avoid $\pi(J0)=J0$, and thus $\pi\Lambda_2$ cannot equal $2\Lambda_1$.

Thus we know $\pi\Lambda_2=\Lambda_2$. The remaining $\pi\Lambda_\ell=\Lambda_\ell$ follow quickly from induction: if $\pi\Lambda_\ell=\Lambda_\ell$ for some $2\leq\ell<r$, then the fusion $\Lambda_1\boxtimes\Lambda_\ell$ tells us $\pi\Lambda_{\ell+1}\in\{\Lambda_{\ell+1},\Lambda_1+\Lambda_\ell\}$. But $h(\Lambda_1+\Lambda_\ell)=(\ell+1)!/\ell$ and $h(\Lambda_{\ell+1})=(\ell+1)!$, so $\pi\Lambda_{\ell+1}=\Lambda_{\ell+1}$. Thus π fixes all fundamental weights, and since these comprise a fusion-generator (see the discussion at the end of §2.2) we know that π must fix everything in P_+.

4.3. The B-series argument

$k=1$ is easy: $P_+=\{0,J0,\Lambda_r\}$ and $\pi=id.$ is automatic. $k=2$ will be done later in this subsection. Assume now that $k\ge 3$.

From Proposition 4.1(b) we can write $\pi\Lambda_1=J^a\Lambda_1$ and $\pi'\Lambda_1=J^{a'}\Lambda_1$. We know $\pi J0=J0$, so (2.7b) says π must take spinors to spinors, and nonspinors to nonspinors. Then we will have $\chi_{\Lambda_1}[\psi]=(-1)^{a'}\chi_{\Lambda_1}[\pi\psi]$ for any spinor ψ. Now if $a'=1$, then π will take the spinors which maximize χ_{Λ_1}, to those which minimize it. Both these maxima and minima can be easily found from (3.2). Thus we get that $\pi(\mathcal{S}\Lambda_r)$ equals $k\Lambda_r$ (when k odd) or $\mathcal{S}((k-1)\Lambda_r)$ (when k even). But the sets $\mathcal{S}\Lambda_r$ and $k\Lambda_r$ have different cardinalities ($k\Lambda_r$ is a J-fixed-point), and so can't get mapped to each other. Also, the fusions $\Lambda_1\boxtimes\Lambda_r=\Lambda_r\boxplus(\Lambda_1+\Lambda_r)$ and $J^a\Lambda_1\boxtimes(J^i(k-1)\Lambda_r)=(J^{a+i}(k-1)\Lambda_r)\boxplus(J^{a+i+1}(k-1)\Lambda_r)\boxplus J^{a+i+1}(\Lambda_{r-1}+(k-3)\Lambda_r)$ have different numbers of weights on their right sides, so also $\pi\Lambda_r\not\in\mathcal{S}(k-1)\Lambda_r$.

Thus $a'=0$ and $\pi\Lambda_r=J^b\Lambda_r$ for some b. Similarly, $a=0$. Hitting π with $\pi[1]^b$, we may assume that π fixes Λ_r.

Now assume π fixes Λ_ℓ, for $1\le\ell<r-1$. Then the fusion $\Lambda_1\boxtimes\Lambda_\ell$ says that $\pi\Lambda_{\ell+1}$ equals $\Lambda_{\ell+1}$ or $\Lambda_1+\Lambda_\ell$. But from (3.2) we find

$$\begin{aligned}\chi_{\Lambda_1}[\Lambda_{\ell+1}]-\chi_{\Lambda_1}[\Lambda_1+\Lambda_\ell]=&\,2\,\{\cos(\pi\frac{2r-2\ell+1}{\kappa})-\cos(\pi\frac{2r-2\ell-1}{\kappa})+\cos(\pi\frac{2r+1}{\kappa})\\&-\cos(\pi\frac{2r+3}{\kappa})\}=4\cos(\pi\frac{2r-\ell+1}{\kappa})\,\{\cos(2\pi\frac{\ell}{\kappa})-\cos(2\pi\frac{\ell+1}{\kappa})\}>0\end{aligned}$$

Hence π will fix $\Lambda_{\ell+1}$ if it fixes Λ_ℓ, concluding the argument.

Now consider the more interesting case: $k=2$. Then $\kappa=2r+1$; recall the weights in $P_+(B_{r,2})$ are the simple-currents $0,J0$, the J-fixed-points $\gamma^1,\ldots,\gamma^r$ (notation defined in §3.2), and the spinors $\Lambda_r,J\Lambda_r$. Because $\pi(J0)=\pi'(J0)=J0$, we know both π and π' must take J-fixed-points to J-fixed-points, i.e. $\pi\Lambda_1=\gamma^m$ and $\pi'\Lambda_1=\gamma^{m'}$ for some $1\le m,m'\le r$. It is easy to compute [25]

$$\frac{S_{\gamma^a\gamma^b}}{S_{0\gamma^b}}=2\cos(2\pi\frac{ab}{\kappa})\ . \tag{4.2}$$

From this we see $m\,m'\equiv\pm1\pmod{\kappa}$, so m is coprime to κ. Hitting it with the Galois fusion-symmetry $\pi\{m'\}$, we see that we may assume $\pi\Lambda_1=\pi'\Lambda_1=\Lambda_1$.

Now use (4.2) to get $\pi\gamma^i=\gamma^i$ for all i. Then π equals the identity or $\pi[1]$, depending on what π does to Λ_r.

4.4. The C-series argument

By rank-level duality, we may take $r\le k$. For now assume $(r,k)\ne(2,3)$. Then we know $\pi\Lambda_1=J^a\Lambda_1$ and $\pi\Lambda_1=J^{a'}\Lambda_1$ for some a,a'. Since $\pi J0=\pi'J0=J0$, (2.7b) says $a=a'=0$ if kr is odd. Since $\chi_{\Lambda_1}[\Lambda_1]>0$ (using (3.3)), $S_{\Lambda_1\Lambda_1}=S_{J^a\Lambda_1,J^{a'}\Lambda_1}$ implies that $a=a'$ also holds when kr is even, and hence we may assume (hitting with $\pi[1]^a$) that also

$a = a' = 0$ holds for kr even. From the fusion $\Lambda_1 \boxtimes \Lambda_\ell$ we get $\pi\Lambda_{\ell+1} \in \{\Lambda_{\ell+1}, \Lambda_1 + \Lambda_\ell\}$ if $\pi\Lambda_\ell = \Lambda_\ell$; for $r < k$ conclude the argument with the calculation

$$\chi_{\Lambda_1}[\Lambda_{\ell+1}] - \chi_{\Lambda_1}[\Lambda_1 + \Lambda_\ell] = 4\cos(\pi\,\frac{2r+2-\ell}{2\kappa})\,\{\cos(\pi\,\frac{\ell}{2\kappa}) - \cos(\pi\,\frac{\ell+2}{2\kappa})\} > 0$$

as in §4.3. When $r = k$, that inequality only holds for $\ell > 1$, but we can force $\pi\Lambda_2 = \Lambda_2$ by hitting π if necessary with π_{rld}.

The remaining case $C_{2,3}$ follows because $\pi' J0 = J0$: by (2.7b) $\pi\Lambda_1 \notin \mathcal{S}\Lambda_2$, and by (2.7a) $\pi\Lambda_1 \neq 3\Lambda_1$ ($3\Lambda_1$ is a J-fixed-point).

4.5. The D-series argument

$k = 1$ is trivial, and $k = 2$ will be considered shortly. For $k > 2$, Proposition 4.1 tells us that $\pi\Lambda_1 = J_v^a J_s^b \Lambda_1$ and $\pi'\Lambda_1 = J_v^{a'} J_s^{b'} \Lambda_1$, for $a, a', b, b' \in \{0, 1\}$. Immediate from (3.4) is that $\chi_{\Lambda_1}[\Lambda_1] > 0$ and that $\chi_{\Lambda_1}[\psi]$, for a spinor ψ, takes its maximum at $C^i J_v^j \Lambda_r$. Our first step is to force $\pi\Lambda_1 = \pi'\Lambda_1 = \Lambda_1$. Unfortunately this requires a case analysis.

Consider first even $r \neq 4$, and even $k > 2$. Now, $0 \neq S_{\Lambda_1\Lambda_1} = S_{\pi\Lambda_1, \pi'\Lambda_1}$ forces $b = b'$; hence hitting with the simple-current automorphism $\pi\begin{bmatrix} 0 & a \\ a' & b \end{bmatrix}$, we may assume $\pi\Lambda_1 = \pi'\Lambda_1 = \Lambda_1$.

Next consider even $r \neq 4$ and odd $k > 2$. Either of $\pi\Lambda_1 = J_v\Lambda_1$ or $\pi'\Lambda_1 = J_v\Lambda_1$ is impossible, by comparing $S_{\Lambda_1, J_s 0}$ and $S_{J_v\Lambda_1, J0}$ for any simple-current J. For any of the three remaining choices of $J_v^a J_s^b \Lambda_1$, we can find a simple-current automorphism of the form $\pi\begin{bmatrix} * & a \\ * & b \end{bmatrix}$; hitting its inverse onto π allows us to take $a = b = 0$. Again $0 \neq S_{\Lambda_1\Lambda_1}$ forces $b' = 0$, and now $a' = 1$ is forbidden. Thus again $\pi\Lambda_1 = \pi'\Lambda_1 = \Lambda_1$.

As usual, $r = 4$ is complicated by triality. We can force $\pi\Lambda_1 = \Lambda_1$. That we can also take $\pi'\Lambda_1 = \Lambda_1$, follows from the inequality $\chi_{\Lambda_1}[\Lambda_1] > \chi_{\Lambda_1}[\Lambda_3] = \chi_{\Lambda_1}[\Lambda_4] > 0$, valid for $k \geq 3$. Establishing that inequality from (3.4) is equivalent to showing

$$1 + \cos(x) + \cos(2x) + \cos(4x) > \cos(x/2) + \cos(3x/2) + \cos(5x/2) + \cos(7x/2)$$

for $0 < x \leq 2\pi/9$, which can be shown e.g. using Taylor series.

For odd r, the charge-conjugation C equals C_1. Since it must commute with π, i.e. that $C_1\pi\Lambda_1 = J_v^{a+b} J_s^b \Lambda_1$ must equal $\pi C_1 \Lambda_1 = J_v^a J_s^b \Lambda_1$, we get that $b = 0$. Similarly $b' = 0$. When k is odd, eliminate $a = 1$ and $a' = 1$ by comparing $S_{\Lambda_1, J_s 0}$ and $S_{J_v\Lambda_1, J0}$ as before. The hardest case is k even. We can force $\pi\Lambda_1 = \Lambda_1$ by hitting with $\pi[a]$. Suppose for contradiction that $\pi'\Lambda_1 = J_v\Lambda_1$. We know $\pi'(J_v 0) = J_v 0$ (compare $S_{\Lambda_1, J_v 0}$ and $S_{\Lambda_1, J0}$), so by (2.7b) $\pi\Lambda_r$ must be a spinor. $\chi_{\Lambda_1}[\Lambda_r] = \chi_{J_v\Lambda_1}[\pi\Lambda_r]$ requires $\pi\Lambda_r = C_1^i J_v^j J_s \Lambda_r$. From the $\Lambda_1 \boxtimes \Lambda_r$ fusion we get $\pi\Lambda_{r-1} = C_1^i J_v^j J_s \Lambda_{r-1}$, but $C\pi = \pi C$ says that $\pi\Lambda_{r-1} = C_1^i J_v^{j+1} J_s \Lambda_{r-1}$ — a contradiction.

Thus in all cases we have $\pi\Lambda_1 = \pi'\Lambda_1 = \Lambda_1$. We know $\pi'(J_v 0) = J_v 0$ (compare $S_{\Lambda_1, J_v 0}$ and $S_{\Lambda_1, J0}$), so $\pi\Lambda_r$ is a spinor and in fact must equal $\pi\Lambda_r = C_1^i J_v^j \Lambda_r$. Hitting with $(C_1^i \pi_v^j)^{-1}$, we can require $\pi\Lambda_r = \Lambda_r$. That $\pi\Lambda_{r-1}$ must now equal Λ_{r-1} follows from the $\Lambda_1 \boxtimes \Lambda_r$ fusion.

Next, note that we know from $\Lambda_1 \boxtimes \Lambda_1$ that $\pi\Lambda_2$ is Λ_2 or $2\Lambda_1$. As in §4.2, the fusion $(2\Lambda_1) \boxtimes \Lambda_1 \boxtimes \cdots \boxtimes \Lambda_1$ ($k-2$ times) contains the simple-current $J_v 0$, but $\Lambda_2 \boxtimes \Lambda_1 \boxtimes \cdots \boxtimes \Lambda_1$ ($k-2$ times) doesn't, and thus $\pi\Lambda_2 = \Lambda_2$.

Assume $\pi\Lambda_\ell = \Lambda_\ell$. Using the fusions $\Lambda_1 \boxtimes \Lambda_\ell$ (for $1 < \ell < r-2$), and noting that

$$\chi_{\Lambda_1}[\Lambda_{\ell+1}] - \chi_{\Lambda_1}[\Lambda_1 + \Lambda_\ell] = 4\cos(\pi\,\frac{2r-\ell}{\kappa})\,\{\cos(\pi\,\frac{\ell}{\kappa}) - \cos(\pi\,\frac{\ell+2}{\kappa})\}$$

equals 0 only when $\ell = r+1-k/2$, we see that $\pi\Lambda_{\ell+1} = \Lambda_{\ell+1}$ except possibly for $\ell = r+1-k/2$ (hence $2r-2 \geq k \geq 4$). For that ℓ, use q-dimensions:

$$\frac{\mathcal{D}(\Lambda_1+\Lambda_\ell)}{\mathcal{D}(\Lambda_{\ell+1})} = \frac{\sin(2\pi\,(k-2)/\kappa)}{\sin(2\pi/\kappa)} > 1\ ,$$

which is valid for these k. So we also know $\pi\Lambda_i = \Lambda_i$ for all $i \leq r-2$, and we are done.

All that remains is $D_{r,2}$. Recall the λ^i defined in §3.4. Note that $\mathcal{D}(\Lambda_r) = \sqrt{r}$, $\mathcal{D}(\lambda^a) = 2$, and $S_{\lambda^a\lambda^b}/S_{0\lambda^b} = 2\cos(\pi ab/r)$. For $r \neq 4$, the q-dimensions force $\pi\Lambda_1 = \lambda^m$ and $\pi'\Lambda_1 = \lambda^{m'}$, and $S_{\Lambda_1\Lambda_1} = S_{\lambda^m\lambda^{m'}}$ says $mm' \equiv \pm 1$ (mod $2r$). So without loss of generality we may take $m = m' = 1$. The rest of the argument is easy.

For $D_{4,2}$, we can force $\pi\Lambda_1 = \Lambda_1$, and then eliminate $\pi'\Lambda_1 = \Lambda_{r-1}$ or Λ_r by $S_{\Lambda_1\Lambda_1} \neq 0 = S_{\Lambda_1\Lambda_r} = S_{\Lambda_1\Lambda_{r-1}}$. The rest of the argument is as before.

4.6. The arguments for the exceptional algebras

The exceptional algebras follow quickly from the fusions (and Dynkin diagram symmetries) given in §§3.5-3.9.

For example, consider $E_6^{(1)}$ for $k \geq 2$. Proposition 4.1 tells us $\pi\Lambda_1 = C^a J^b \Lambda_1$ for some a, b, and we know $\pi' J0 = J^c 0$ for $c = \pm 1$. Hence from (2.7b) we get $kb \not\equiv -1$ (mod 3). Hitting π with $\pi[-b]^{-1}C^a$, we need consider only $\pi\Lambda_1 = \Lambda_1$. It is now immediate that $\pi\Lambda_5 = \Lambda_5$, by commuting π with C. From (3.6a) we get that π must permute Λ_2 and $2\Lambda_1$. Compare (3.6c) with (3.6d): since for any $k \geq 2$ they have different numbers of summands, we find in fact that π will fix both Λ_2 (hence Λ_4) and $2\Lambda_1$. From (3.6b) we get that π permutes Λ_6 and $\Lambda_1 + \Lambda_5$, and so (3.6d) now tells us $\pi\Lambda_6 = \Lambda_6$. Finally, (3.6c) implies (for $k \geq 3$) $\pi\Lambda_3 = \Lambda_3$ (since $C\pi = \pi C$), and we are done for $k \geq 3$. Since $\{\Lambda_1, \Lambda_2, \Lambda_4, \Lambda_5, \Lambda_6\}$ is a fusion-generator for $k = 2$ (see §2.2), we are also done for $k = 2$.

For $E_8^{(1)}$ when $k \geq 7$, (3.7a) tells us that $\Lambda_2, \Lambda_7, 2\Lambda_1$ are permuted. For those k, the highest multiplicities in (3.7b)–(3.7d) are 3, 1, 2, respectively, so $\Lambda_2, \Lambda_7, 2\Lambda_1$ must all be fixed. The fusion product (3.7c) also tells us that $\Lambda_3, \Lambda_6, \Lambda_8, \Lambda_1 + \Lambda_7, 2\Lambda_7$ are permuted; (3.7d) then says that the sets $\{\Lambda_6, \Lambda_8\}$, $\{\Lambda_3, \Lambda_1 + \Lambda_7, 2\Lambda_7\}$, and $\{2\Lambda_2, \Lambda_2 + \Lambda_7, 3\Lambda_1, 2\Lambda_1 + \Lambda_2, 2\Lambda_1 + \Lambda_7, 4\Lambda_1\}$ are stabilised. Now (3.7b) implies $\Lambda_3, \Lambda_6, \Lambda_8, 2\Lambda_7$ are all fixed, while the set $\{\Lambda_4, \Lambda_1 + \Lambda_3\}$ is stabilised. Comparing (3.7e) and (3.7f), we get that Λ_4 is fixed and $\Lambda_5, \Lambda_7 + \Lambda_8$ are permuted. Finally, (3.7g) shows Λ_5 also is fixed. To do $E_8^{(1)}$ when $k \leq 6$, knowing q-dimensions really simplifies things.

5. Affine fusion ring isomorphisms

We conclude the paper with the determination of all isomorphisms among the affine fusion rings $\mathcal{R}(X_{r,k})$. Recall Definition 2.1 and the discussion in §2.2.

THEOREM 5.1. *The complete list of fusion ring isomorphisms $\mathcal{R}(X_{r,k}) \cong \mathcal{R}(Y_{s,m})$ when $X_{r,k} \neq Y_{s,m}$ (where X_r, Y_s are simple) is:*
rank-level duality $\mathcal{R}(C_{r,k}) \cong \mathcal{R}(C_{k,r})$ for all r, k, as well as $\mathcal{R}(A_{1,k}) \cong \mathcal{R}(C_{k,1})$;
$\mathcal{R}(B_{r,1}) \cong \mathcal{R}(A_{1,2}) \cong \mathcal{R}(C_{2,1}) \cong \mathcal{R}(E_{8,2})$ for all $r \geq 3$;
$\mathcal{R}(A_{3,1}) \cong \mathcal{R}(D_{odd,1})$;
$\mathcal{R}(D_{r,1}) \cong \mathcal{R}(D_{s,1})$ whenever $r \equiv s \pmod 2$;
$\mathcal{R}(A_{2,1}) \cong \mathcal{R}(E_{6,1})$ and $\mathcal{R}(A_{1,1}) \cong \mathcal{R}(E_{7,1})$;
$\mathcal{R}(F_{4,1}) \cong \mathcal{R}(G_{2,1})$, $\mathcal{R}(F_{4,2}) \cong \mathcal{R}(E_{8,3})$, and $\mathcal{R}(F_{4,3}) \cong \mathcal{R}(G_{2,4})$.

The isomorphism $\mathcal{R}(A_{1,k}) \cong \mathcal{R}(C_{k,1})$ takes $a\Lambda_1$ to $\widetilde{\Lambda}_a$. The isomorphism $\mathcal{R}(F_{4,2}) \cong \mathcal{R}(E_{8,3})$ was found in [14]; it relates $\Lambda_1 \leftrightarrow \widetilde{\Lambda}_8$, $2\Lambda_4 \leftrightarrow \widetilde{\Lambda}_2$, $\Lambda_3 \leftrightarrow \widetilde{\Lambda}_1$, $\Lambda_4 \leftrightarrow \widetilde{\Lambda}_7$. The isomorphism $\mathcal{R}(F_{4,3}) \cong \mathcal{R}(G_{2,4})$ was found in [34,14]; a correspondence which works is $\Lambda_4 \leftrightarrow \widetilde{\Lambda}_1$, $\Lambda_1 \leftrightarrow 2\widetilde{\Lambda}_1$, $\Lambda_3 \leftrightarrow 3\widetilde{\Lambda}_2$, $2\Lambda_4 \leftrightarrow 2\widetilde{\Lambda}_2$, $\Lambda_1 + \Lambda_4 \leftrightarrow \widetilde{\Lambda}_1 + 2\widetilde{\Lambda}_2$, $\Lambda_2 \leftrightarrow 4\widetilde{\Lambda}_2$, $3\Lambda_4 \leftrightarrow \widetilde{\Lambda}_2$, and $\Lambda_3 + \Lambda_4 \leftrightarrow \widetilde{\Lambda}_1 + \widetilde{\Lambda}_2$.

We will sketch the proof here. The idea is to compare invariants for the various fusion rings, case by case. For example, suppose $\mathcal{R}(A_{r,k})$ and $\mathcal{R}(A_{s,m})$ are isomorphic. Then their simple-current groups $\mathbb{Z}_{r+1}$ and $\mathbb{Z}_{s+1}$ must be isomorphic (since simple-currents must get mapped to simple-currents), so $r = s$. Now compare the numbers $\|P_+\|$ of highest-weights: $\binom{r+k}{r} = \binom{r+m}{r}$, which forces $m = k$.

It is also quite useful here to know those weights with second smallest q-dimension. This is a by-product of the proof of Proposition 4.1, and the complete answer is given in [18, Table 3]. Here we will simply state that those weights in $P_+^k(X_r^{(1)})$ with second smallest q-dimension are precisely the orbit $\mathcal{S}\Lambda_\star$, except for: $A_{r,1}$; $B_{r,k}$ for $k \leq 3$; $C_{2,2}, C_{2,3}, C_{3,2}$; $D_{r,k}$ for $k \leq 2$; $E_{6,k}$ for $k \leq 2$; and $E_{7,k}, E_{8,k}, F_{4,k}, G_{2,k}$ for $k \leq 4$.

$C_{r,k}$ and $B_{s,m}$ both have two simple-currents, but their fusion rings can't be isomorphic (generically) because the orbit $J^i\Lambda_1$ has the second smallest q-dimension for both algebras at generic rank/level, but the numbers $Q_j(J^i\Lambda_1)$ for the two algebras are different.

Another useful invariant involves the set of integers ℓ coprime to κN for which $0^{(\ell)}$ is a simple-current. For the classical algebras this is easy to find, using (2.1c): Up to a sign, the q-dimension of $0^{(\ell)}$ (ℓ coprime to 2κ) for the algebras $B_r^{(1)}, C_r^{(1)}, D_r^{(1)}$ is, respectively,

$$\prod_{a=0}^{r-1} \frac{\sin(\pi\ell\,(2a+1)/2\kappa)}{\sin(\pi\,(2a+1)/2\kappa)} \prod_{b=1}^{2r-2} \frac{\sin(\pi\ell b/\kappa)^{[\frac{2r-b}{2}]}}{\sin(\pi b/\kappa)^{[\frac{2r-b}{2}]}} \;,$$

$$\prod_{a=1}^{r-1} \frac{\sin(\pi\ell a/\kappa)^{r-a}\,\sin(\pi\ell\,(2a-1)/2\kappa)^{r-a}}{\sin(\pi a/\kappa)^{r-a}\,\sin(\pi\,(2a-1)/2\kappa)^{r-a}} \prod_{b=r}^{2r-1} \frac{\sin(\pi\ell b/2\kappa)}{\sin(\pi b/2\kappa)} \;,$$

$$\prod_{a=1}^{r-1} \frac{\sin(\pi\ell a/\kappa)^{[\frac{2r-a+1}{2}]}}{\sin(\pi a/\kappa)^{[\frac{2r-a+1}{2}]}} \prod_{b=r}^{2r-3} \frac{\sin(\pi\ell b/\kappa)^{[\frac{2r-b-1}{2}]}}{\sin(\pi b/\kappa)^{[\frac{2r-b-1}{2}]}} \;,$$

where $[x]$ here denotes the greatest integer not more than x. The absolute value of each of these is quickly seen to be greater than 1 unless $\ell \equiv \pm 1 \pmod{2\kappa}$, except for the orthogonal algebras when $k \leq 2$. An isomorphism $\mathcal{R}(X_{r,k}) \cong \mathcal{R}(X_{r',k'})$ would require then that whenever $\ell \equiv \pm 1 \pmod{2\kappa}$ is coprime to κ', it must also satisfy $\ell \equiv \pm 1$ (mod $2\kappa'$), and conversely. This forces $\kappa = \kappa'$, for $X = B$ or D and $k > 2$, or $X = C$ and any k.

If $\mathcal{R}(C_{r,k}) \cong \mathcal{R}(C_{s,m})$, then that Galois argument implies $r + k + 1 = s + m + 1$, so compare numbers of highest-weights: $\binom{r+k}{r} = \binom{r+k}{s}$.

A similar argument works for the orthogonal algebras. For instance suppose $\mathcal{R}(B_{r,k}) \cong \mathcal{R}(B_{s,m})$ but $B_{r,k} \neq B_{s,m}$, and that $k, m > 2$. Then Galois implies $2r + k = 2s + m$. Comparing the value of $\mathcal{D}(\Lambda_1)$ (the second smallest q-dimension when $k > 3$), using (3.2) with $\lambda = 0$, tells us that $2s + 1 = k, 2r + 1 = m$. Now count the number of fixed-points of J in both cases: $\binom{\kappa/2-1}{r-1} = \binom{\kappa/2-1}{s-1}$, i.e. $s - 1 = (k-1)/2$, a contradiction.

For comparing classical algebras with exceptional algebras, a useful device is to count the number of weights appearing in the fusion $\Lambda_\star \boxtimes \Lambda_\star$ (when $\Lambda_\star$ has second smallest q-dimension). For example, for $A_{1,k}$ $(k > 1)$, $C_{r,k}$ ($k > 1$, except for $C_{2,2}, C_{2,3}, C_{3,2}$), and $E_{7,k}$ $(k > 4)$, we learned in §3 that this number is 2, 3, 4 respectively, so none of these can be isomorphic.

For the orthogonal algebras at level 2, useful is the number of weights with second smallest q-dimension (respectively r and $r - 1$ for $B_{r,2}$ and $D_{r,2}$, except for $D_{4,2}$).

For the exceptional algebras, comparing $\mathcal{D}(\Lambda_\star)$ and the number of highest-weights is effective. Recall that both $\|P_+\|$ and $\mathcal{D}(\Lambda_\star)$ for a fixed algebra monotonically increase with k to (respectively) ∞ and the Weyl dimension of $\Lambda_\star$, which is 7, 26, and 248 for G_2, F_4, E_8 respectively. For $E_{8,k}$, $\mathcal{D}(\Lambda_1)$ exceeds 7 for $k \geq 5$, and exceeds 26 for $k \geq 11$, while $F_{4,k}$ exceeds 7 for $k \geq 4$. The number of highest-weights of $E_{8,4}, E_{8,10}$, and $F_{4,3}$ are 10, 135, and 9, so only a small number of possibilities need be considered.

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